A Boundary Control Problem Associated to the Rayleigh-Bénard-Marangoni System

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Abstract

In this paper, we study a boundary control problem associated to the stationary Rayleigh-Bénard-Marangoni (RBM) system in presence of controls for the velocity and the temperature on parts of the boundary. We analyze the existence, uniqueness and regularity of weak solutions for the stationary RBM system in polyhedral domains of $\mathbb{R}^3$, and then, we prove the existence of the optimal solution. By using the Theorem of Lagrange multipliers, we derive an optimality system. We also give a second-order sufficient optimality condition and we establish a result of uniqueness of the optimal solution.

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1 Introduction

Fluid movement by temperature gradients, also called thermal convection, is an important process in nature. Its main applications appear in industry, as for instance, in the growth of semiconductor crystals, but also, thermal convection is the basis for the interpretation of several natural phenomena such as the movement of the earth's plates, the solar activity, large scale circulations of the oceans, movement in the atmosphere, among others. A model of particular interest consists of a horizontal layer of a fluid in a container heated uniformly from below, with the bottom surface and the lateral walls rigid and the upper surface open to the atmosphere. Due to heating, the fluid in the bottom surface expands and it becomes lighter than the fluid in the upper surface, so that, by effect of the buoyancy, the liquid is potentially unstable. Because of the instability, the fluid tends to redistribute. However, this natural tendency will be controlled by its own viscosity. On the other hand, the upper surface, which is free to the atmosphere, experiences changes in its surface tension as a result of the temperature gradients in the surface. Then, it is expected that the temperature gradient exceeds a critical value, above which the instability can manifest.

The first experiments to demonstrate the beginning of Thermal instability in fluids were developed by Henri Bénard in 1900 (see [5]). In his experiments, Bénard considered a very thin layer of liquid, around $1\,mm$ of depth, in a metal plate maintained at a constant temperature. The upper surface was usually free and it was in contact with the air, which was at a lower temperature. Bénard experimented with a variety of liquids with different physical characteristics, mainly interested in the effect of viscosity on the convection, using liquids of high viscosity like wax whale melted and paraffin. In all these cases, Bénard found that when the temperature of the plate gradually increased, at a certain moment, the layer lost stability and formed patterns of hexagonal cells, all alike and correctly aligned.

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A first theoretical interpretation of thermal convection was provided by Lord Rayleigh in 1916 (see [37]), whose analysis was inspired by Bénard's experiment. Rayleigh assumed that the fluid was confined between two horizontal thermally conductive plates and the fluid was being heated from below. Rayleigh considered that the effect of buoyancy is the only one responsible of the beginning of the instability, and theoretically, the results coincided with the reported by Bénard, giving the impression that his model was correct. However, it is known now that Rayleigh's theory is not adequate for explaining the convective mechanism observed by Bénard. In fact, in Bénard's experiments, the free surface was in contact with the atmosphere which generates a surface tension, and Rayleigh, using a plate in the upper surface, eliminated the surface tension's effects.

It should be noted that the surface tension is not constant and it may depend on the temperature or contaminants in the surface. This dependence is called *capillarity* or *Marangoni effect* [26, 31]. The importance of the Marangoni effect in Bénard's experiments was established by Block in [6] from an experimental point of view, and by Pearson [36] from a theoretical point of view. Now is recognized that the Marangoni effect is the main cause of instability and convection in the Bénard original experiments.

For the foregoing reasons, we consider the physical situation of a horizontal layer of a fluid in a cubic container of height d (x_3-coordinate), of length L_1 (x_2-coordinate) and width l_1 (x_1-coordinate). The bottom surface of the container and the lateral walls are rigid and the upper surface is open to the atmosphere. In order to describe the system, we use the Oberbeck-Boussinesq approximation [7], which assumes that the thermodynamical coefficients are constant, except in the case of the density in the buoyancy term, which is considered as being $\rho_0\,[1-\alpha\,(\theta-\theta_a)]$. Here ρ_0 is the mean density, θ_a is the temperature of the environment and α is the thermal expansion coefficient, which is positive for most liquids. Moreover, we assume that the surface tension is a function of the temperature, and it is approximated by $\sigma=\sigma_0-\gamma\,(\theta-\theta_a)$. Here, σ_0 is the surface tension at temperature θ_a, and γ is the ratio of change of the surface tension with the temperature (γ is positive for the more commonly used liquids). Also, the free surface is presumed not to be distorted, that is, the vertical component of the velocity in the free surface always will be zero. Then, we consider that the domain, in which the fluid is confined, is given by $\Omega=(0,\,l_1)\times(0,\,L_1)\times(0,\,d)$. However, the analysis developed in this paper allows us to consider a domain Ω with more general geometries, specifically, we can consider $\Omega=\hat{\Omega}\times(0,d)$, being $\hat{\Omega}$ a Lipschitz bounded domain of $\mathbb{R}^2$.

In stationary regime, the RBM system is given by the following coupling between the Navier-Stokes equations and heat equation:

$$\left\{\begin{aligned} \rho_0\,(\mathbf{u}\cdot\nabla)\,\mathbf{u}+\nabla p-\mu\Delta\mathbf{u}=\rho_0\,[1-\alpha\,(\theta-\theta_a)]\,\vec{g} &\ \text{ in } \Omega,\\ \rho_0\hat{C}_p\,(\mathbf{u}\cdot\nabla)\,\theta=K\Delta\theta &\ \text{ in } \Omega,\\ \operatorname{div}\mathbf{u}=0 &\ \text{ in } \Omega, \end{aligned}\right. \tag{1.1}$$

where the unknowns are $\mathbf{u}(\mathbf{x})=(u_1(\mathbf{x}),u_2(\mathbf{x}),u_3(\mathbf{x}))\in\mathbb{R}^3$, $\theta(\mathbf{x})\in\mathbb{R}$ and $p(\mathbf{x})\in\mathbb{R}$, which represent the velocity field, the temperature and the hydrostatic pressure of the fluid, respectively, at a point $\mathbf{x}\in\Omega$. The constant $\hat{C}_p$ is the heat capacity per unit mass of the fluid, μ its viscosity, K its thermal conductivity and the field $\vec{g}$ is the acceleration due to gravity.

In order to express the system in adimensional form, we make the following changes of variables:

$$x_1'=\frac{x_1}{d},\ x_2'=\frac{x_2}{d},\ x_3'=\frac{x_3}{d},\ u_1'=\frac{du_1}{\kappa},\ u_2'=\frac{du_2}{\kappa},\ u_3'=\frac{du_3}{\kappa},\ \theta'=\frac{\theta-\theta_a}{\theta_u},\ p'=\frac{d^2p}{\rho_0\nu\kappa},$$

where $\theta_u=\theta_c-\theta_a$ with θ_c the temperature at the bottom plate, $\kappa=\dfrac{K}{\rho_0\hat{C}_p}$ and $\nu=\dfrac{\mu}{\rho_0}$. Thus, removing

the primes to simplify the notation, from (1.1) we get

$$\begin{cases} (\mathbf{u}\cdot\nabla)\mathbf{u} = Pr\,[(b+R\theta)\,\vec{e}_3 - \nabla p + \Delta\mathbf{u}] & \text{in } \Omega,\\ (\mathbf{u}\cdot\nabla)\theta = \Delta\theta & \text{in } \Omega,\\ \operatorname{div}\mathbf{u} = 0 & \text{in } \Omega, \end{cases} \tag{1.2}$$

with $\Omega = (0,\,l)\times(0,\,L)\times(0,\,1)$, where $l = \frac{l_1}{d}$ and $L = \frac{L_1}{d}$. Moreover, $Pr = \dfrac{\nu}{\kappa}$, $R = \dfrac{|\vec{g}|\alpha\theta_u d^3}{\kappa\nu}$ and $b = -\dfrac{|\vec{g}|d^3}{\kappa\nu}$. The number R is known as the Rayleigh number and it measures the effect of buoyancy; Pr is known as the Prandtl number and it represents the relationship between the speed of diffusion of momentum and the rate of diffusion of heat in the fluid, and $\vec{e}_3$ is the unit vector in the third direction, that is, $\vec{e}_3 = (0,0,1)$.

Let us denote by $\partial\Omega$ the boundary of Ω and let $\Gamma_1 := \partial\Omega\cap\{x_3 = 1\}$ and $\Gamma_0 := \partial\Omega\setminus\Gamma_1$. Then, the following boundary conditions are imposed:

$$\begin{cases} \mathbf{u} = \mathbf{0} \ \text{ on } \Gamma_0, \qquad u_3 = 0 \ \text{ on } \Gamma_1,\\ \frac{\partial u_i}{\partial\mathbf{n}} + M\frac{\partial\theta}{\partial x_i} = 0, \ i = 1,2, \ \text{ on } \Gamma_1,\\ \frac{\partial\theta}{\partial\mathbf{n}} + B\theta = 0 \ \text{ on } \Gamma_1, \qquad \theta = \theta_c \ \text{ on } \{x_3 = 0\},\\ \frac{\partial\theta}{\partial\mathbf{n}} = 0 \ \text{ on } \Gamma_0\setminus\{x_3 = 0\}, \end{cases} \tag{1.3}$$

where $\mathbf{n} = (n_1, n_2, n_3)$ is the normal vector pointing outward, $M = \frac{\gamma\theta_u d}{\rho_0\nu k}$, $B = \frac{hd}{K} > 0$, and h is the heat exchange coefficient of the surface with the atmosphere.

The boundary conditions for the velocity in $(1.3)_1$ are no slip conditions on the rigid and free surface. The condition $(1.3)_2$ takes into account the Marangoni effect, which represents the mass transfer at an interface between two fluids due to a surface tension gradient. Conditions $(1.3)_3$ say that on the lateral surfaces there is not heat flow (adiabatics), that on the free surface is allowed the heat flow, and that the bottom surface is maintained at temperature θ_c (isothermal).

From the point of view of the existence of solution of RBM problem, recently in [35] was discussed a bifurcation problem in which, considering either the Rayleigh number or the Prandtl number as bifurcation parameters. By using the local bifurcation theory due to Crandall and Rabinowitz [8], the authors showed the existence of stationary solutions to the problem (1.2)-(1.3), which bifurcate from a basic state of heat conduction. For basic state we refer to the exact solution of the problem (1.2)-(1.3), which is given by

$$\mathbf{u}_b = \mathbf{0}, \quad \theta_b = \theta_c - \frac{\theta_c B}{1+B}x_3 \ \text{ and } \ p_b = p_1 x_3 + p_2 x_3^2. \tag{1.4}$$

Previously to [35], in [9, 21, 22, 25] were obtained numerical results on the existence of solutions that bifurcate of the basic stationary states, instability and patter formation problems, as well as a validation of initial and boundary conditions. However, from a theoretical point of view, no more results are available in the literature. The main difficulty in the treating the RBM problem (1.2)-(1.3), beyond the coupling between Navier-Stokes system and heat equation, are the crossed boundary conditions (1.3) involving tangential derivatives of the temperature and normal derivatives of the velocity field; in fact, in order to define tangential derivatives at the boundary, intended in the trace sense, it is necessary regularity of the weak solutions; this fact involves the geometry of the domain in order to use elliptic regularity in Sobolev spaces $W^{k,p}$ for the Laplace and Stokes equations, in polyhedral domains (see [11, 12, 16] for elliptic regularity results associated to Laplace equation and [10, 24, 33, 34] for elliptic regularity results related to the Stokes equation; see also [38] for related problems associated to the Boussinesq system).

From the point of view of optimal control theory, unlike to the Navier-Stokes stationary equations, results on boundary control problems in which the cost functional is subject to state equations governed

by RBM system are not known. Some optimal control results associated with the Boussinesq equations are known, see for example [1, 2, 3, 4, 27, 28, 29]; however, the mathematical formulation and the boundary control problem for Boussinesq equations differ from the RBM model in the type of boundary conditions, principally the condition $(1.3)_2$ which takes into account the Marangoni effect, as well as the dimension of the domain. More exactly, in [1] the authors studied an optimal control problem minimizing the turbulence caused by the heat convection. The states are given by 3D-Boussinesq equations with Neumann control on the temperature. In [2, 3, 4], the authors analyzed optimal control problems for the 3D-Boussinesq equations with Neumann and Dirichlet boundary controls. The results of [35] do not include control theory. In [27, 29] the authors analyzed optimal control problems associated to the 2D-Boussinesq equations. The controls considered may be of either the distributed or the Neumann type. In [29] the author considers the approximation, by finite element methods, of the optimality system and derive optimal error estimates. The convergence of a simple gradient method is proved and some numerical results are given. In [28] the authors studied an optimal control problem for the Boussinesq equations, also in 2D, with Dirichlet control on the temperature. A gradient method for the solution of the discrete optimal control problem is presented and analyzed. Finally, the results of some computational experiments are presented. In the previous references, sufficient optimality conditions were not analyzed. Optimal control problems for the Navier-Stokes equations through the action of Dirichlet boundary conditions have been analyzed (see for instance in [14, 19, 20]). In some cases, numerical results, either by solving the optimality system or by optimization methods, have been obtained.

In this paper, we analyze an optimal control problem for which the velocity and the temperature of the fluid are controlled by boundary data along portions of the boundary; the cost functional is given by a sum of functionals which measure, in the L^2-norm, the difference between the velocity (respectively, the temperature) and a given prescribed velocity (respectively, a prescribed temperature). The cost functional also measures the vorticity of the flow. The fluid motion is constrained to satisfy the stationary system of RBM. The exact mathematical formulation will be given in Section 2. We will prove the solvability of the optimal control problem and, by using the Lagrange multiplier method, we state the first-order optimality conditions; we derive an optimality system and give a second-order sufficient optimality condition. Moreover, we also study the uniqueness of the optimal solution. Beside to the solvability of the optimal control problem, we first prove the existence of weak solutions for RBM system with nonhomogeneous boundary data, as well as the uniqueness and regularity properties. It is worthwhile to remark that the proof of existence and regularity of weak solutions for RBM system is not a simple generalization of the similar ones to deal with Navier-Stokes or related models in fluid mechanics [39]. If fact, we are considering non homogeneous crossed boundary conditions involving tangential derivatives of the temperature and normal derivatives of the velocity field, which permit to deal with pointwise constrained boundary optimal control of Dirichlet and Neumann type. In [35], the boundary conditions are homogeneous, and thus, boundary control problems are not considered. The non homogeneous boundary conditions are assumed in spaces of kind $H^{1/2}_{00}(\Gamma), \Gamma \subseteq \partial\Omega$, which are natural from the variational point of view; these space, which are used as control spaces, are closed subspaces of $H^{1/2}(\Gamma)$ and satisfy the embeddings $H^1(\Gamma) \hookrightarrow H^{1/2}_{00}(\Gamma) \hookrightarrow L^2(\Gamma)$. On the other hand, to define tangential derivatives at the boundary, intended in the trace sense, it is necessary to analyze the regularity of the weak solutions, in particular, it is required the H^2-regularity for the temperature (cf. $(1.3)_2$ and Lemma 3.2 below). However, due the geometry of the domain, the regularity of the weak solutions, when non homogeneous boundary conditions are assumed, is a nontrivial subject. For that, we adapt regularity results for the Laplace equation with Dirichlet-Neumann boundary homogeneous conditions in corner domains of [11, 12, 16], and some ideas of [23] to treat the Robin and Neumann nonhomogeneous boundary conditions. On the other hand, from the point of view of the control theory, as far as we known, our results are the first ones dealing with with pointwise constrained boundary optimal control of the RBM system, by using spaces $H^{1/2}_{00}(\Gamma)$ as the control spaces. We give necessary and sufficient optimality conditions which are a significant advance in the analysis of controlling these equations. In order to obtain necessary optimality conditions we use an approach which differs from the other ones in the case of 3D-Boussinesq equations (cf. [1, 2]). In fact, in order to derive the optimality

conditions, in [2] the author used a theorem of Ioffe and Tikhomorov and also he assumed a property, called *Property C*, whereas that in [1], the authors used a penalization method because in that case the relation control-state is multivalued. It is worthwhile to remark that in the previous references related to convection problems, sufficient optimality conditions were not analyzed. Finally, from the point of view of numerical results, since the analysis of the control problem yields variational inequalities as optimality conditions, the numerical analysis offers new challenges, for instance, the applicability of the semi-smooth Newton method in order to obtain a numerical solution (cf. [14] for numerical results in the context of Navier-Stokes model).

The outline of this paper is as follows: In Section 2, we give a precise definition of the optimal control problem to be studied and, in Section 3, we prove the existence and uniqueness of weak solutions, as well as we show regularity properties. In Section 4, we prove the existence of the optimal solution. In section 5, we obtain the first-order optimality conditions, and by using the Lagrange multipliers method we derive an optimality system. In Section 6, we give a second-order sufficient optimality condition. In Section 7, we establish a result of uniqueness of the optimal solution.

2 Statement of the boundary control problem

Throughout this paper we use the Sobolev space $H^m(\Omega)$, and $L^p(\Omega)$, $1 \leq p \leq \infty$, with the usual notations for norms $\|\cdot\|_{H^m(\Omega)}$ and $\|\cdot\|_{L^p(\Omega)}$ respectively. If H is a Hilbert space we denote its inner product by $(\cdot,\cdot)_H$; in particular, the $L^2(\Omega)$-inner product will be represented by $(\cdot,\cdot)_{L^2(\Omega)}$. If X is a general Banach space, its topological dual will be denoted by X' and the duality product by $\langle\cdot,\cdot\rangle_{X',X}$. Corresponding Sobolev spaces of vector valued functions will be denoted by $\mathbf{H}^1(\Omega)$, $\mathbf{H}^2(\Omega)$, $\mathbf{L}^2(\Omega)$, and so on. If Γ is a connected subset of the boundary $\partial\Omega$, we define the trace space

$$H_{00}^{1/2}(\Gamma) := \{v \in L^2(\Gamma) : \text{ there exists } g \in H^{\frac{1}{2}}(\partial\Omega),\ g|_{\Gamma} = v,\ g|_{\partial\Omega\setminus\Gamma} = 0\},$$

which is a closed subspace of $H^{\frac{1}{2}}(\Gamma)$ (see [13], p. 397), where $H^{\frac{1}{2}}(\Gamma)$ is the restriction of the elements of $H^{\frac{1}{2}}(\partial\Omega)$ to Γ. We also will use the following Banach spaces

$$\begin{aligned}
&\mathbf{X} := \{\mathbf{u} = (u_1,u_2,u_3) \in \mathbf{H}^1(\Omega) : \text{ div } \mathbf{u} = 0,\ u_3 = 0 \text{ on } \Gamma_1\},\\
&\mathbf{X}_0 := \{\mathbf{u} \in \mathbf{H}^1(\Omega) : \text{ div } \mathbf{u} = 0, \mathbf{u} = 0 \text{ on } \Gamma_0,\ u_3 = 0 \text{ on } \Gamma_1\},\\
&Y := \{S \in H^1(\Omega) : S = 0 \text{ on } \{x_3 = 0\}\},\\
&\widetilde{\mathbf{X}} := \{\mathbf{u} \in \mathbf{X} : \mathbf{u}\cdot\mathbf{n} = 0 \text{ on } \Gamma_0 \setminus \{x_3 = 0\}\},\\
&\widetilde{\mathbf{H}}_{00}^{1/2}(\Gamma) := \left\{\mathbf{v} \in \mathbf{H}_{00}^{1/2}(\Gamma) : \int_\Gamma \mathbf{v}\cdot\mathbf{n} = 0,\ \mathbf{v}\cdot\mathbf{n} = 0 \text{ on } \Gamma \setminus \{x_3 = 0\}\right\}.
\end{aligned}$$

Moreover, if Γ is an arbitrary subset of $\partial\Omega$, we use the notation $\langle f, g\rangle_\Gamma$ to represent the integral $\int_\Gamma fg\, dS$. In the paper, the letter C will denote diverse positive constants which may change from line to line or even within a same line.

In order to establish the boundary control problem, we consider the following stationary model related to (1.2)-(1.3) with nonhomogeneous boundary data:

$$\left\{\begin{aligned}
(\mathbf{u}\cdot\nabla)\mathbf{u} = Pr\,[(b + R\theta)\,\vec{e}_3 - \nabla p + \Delta\mathbf{u}] \quad &\text{in } \Omega,\\
(\mathbf{u}\cdot\nabla)\theta = \Delta\theta \quad &\text{in } \Omega,\\
\text{div}\,\mathbf{u} = 0 \quad &\text{in } \Omega,\\
\mathbf{u} = \mathbf{g} \text{ on } \Gamma_0^1, \qquad \mathbf{u} = \mathbf{u}^0 \quad &\text{on } \Gamma_0^2,\\
u_3 = 0 \quad &\text{on } \Gamma_1,\\
\frac{\partial u_i}{\partial\mathbf{n}} + M\frac{\partial\theta}{\partial x_i} = 0,\ i = 1,2, \quad &\text{on } \Gamma_1,\\
\frac{\partial\theta}{\partial\mathbf{n}} + B\theta = 0 \text{ on } \Gamma_1, \qquad \theta = \phi_2 \quad &\text{on } \{x_3 = 0\},\\
\frac{\partial\theta}{\partial\mathbf{n}} = \phi_1 \quad &\text{on } \Gamma_0 \setminus \{x_3 = 0\},
\end{aligned}\right. \tag{2.1}$$

where $\Gamma_0 = \Gamma_0^1 \cup \Gamma_0^2$ with $\Gamma_0^1 \cap \Gamma_0^2 = \emptyset$, the vector $\mathbf{u}^0 = (u_1^0, u_2^0, u_3^0) \in \widetilde{\mathbf{H}}_{00}^{1/2}(\Gamma_0^2)$ is a Dirichlet condition for the velocity $\mathbf{u}$ on Γ_0^2; the field $\mathbf{g} = (g_1, g_2, g_3) \in \widetilde{\mathbf{H}}_{00}^{1/2}(\Gamma_0^1)$ is given and denotes a control for $\mathbf{u}$ on Γ_0^1; additionally, $\phi_1 \in H^{\frac{1}{2}}(\Gamma_0 \setminus \{x_3 = 0\})$ is a given function which denotes a Neumman control to temperature θ on $\Gamma_0 \setminus \{x_3 = 0\}$, and $\phi_2 \in H_{00}^{1/2}(\{x_3 = 0\})$ is a Dirichlet control to temperature θ on $\{x_3 = 0\}$.

Suppose that $\mathbf{U}_1 \subset \widetilde{\mathbf{H}}_{00}^{1/2}(\Gamma_0^1)$, $\mathcal{U}_2 \subset H^{\frac{1}{2}}(\Gamma_0 \setminus \{x_3 = 0\})$ and $\mathcal{U}_3 \subset H_{00}^{1/2}(\{x_3 = 0\})$ are nonempty sets, and γ_i, $i = 1, ..., 6$, are constants. Assume one of the following conditions:

(i) $\gamma_i \geq 0$ for $i = 1, 2, ..., 6$, with $\gamma_1, \gamma_2, \gamma_3$ not simultaneously zero, and $\mathbf{U}_1$, $\mathcal{U}_2$ and $\mathcal{U}_3$ are bounded closed convex sets;

(ii) $\gamma_i \geq 0$ for $i = 1, 2, 3$, with $\gamma_1, \gamma_2, \gamma_3$ not simultaneously zero, $\gamma_i > 0$ for $i = 4, 5, 6$ and $\mathbf{U}_1$, $\mathcal{U}_2$ and $\mathcal{U}_3$ are closed convex sets.

We study the following constrained minimization problem on weak solutions to problem (2.1), for fixed data $\mathbf{u}^0 \in \widetilde{\mathbf{H}}_{00}^{1/2}(\Gamma_0^2)$:

$$\left\{\begin{array}{l} \text{Find } [\mathbf{u}, \theta, \mathbf{g}, \phi_1, \phi_2] \in \widetilde{\mathbf{X}} \times H^1(\Omega) \times \mathbf{U}_1 \times \mathcal{U}_2 \times \mathcal{U}_3 \text{ such that the functional} \\ \mathcal{J}[\mathbf{u}, \theta, \mathbf{g}, \phi_1, \phi_2] = \frac{\gamma_1}{2}\|\text{rot } \mathbf{u}\|_{L^2(\Omega)}^2 + \frac{\gamma_2}{2}\|\mathbf{u} - \mathbf{u}_d\|_{L^2(\Omega)}^2 + \frac{\gamma_3}{2}\|\theta - \theta_d\|_{L^2(\Omega)}^2 + \frac{\gamma_4}{2}\|\mathbf{g}\|_{H^{\frac{1}{2}}(\Gamma_0^1)}^2 \\ \qquad\qquad + \frac{\gamma_5}{2}\|\phi_1\|_{H^{\frac{1}{2}}(\Gamma_0 \setminus \{x_3 = 0\})}^2 + \frac{\gamma_6}{2}\|\phi_2\|_{H^{\frac{1}{2}}(\{x_3 = 0\})}^2, \\ \text{is minimized subject to the constraint that } [\mathbf{u}, \theta] \text{ is a weak solution of (2.1). Here } \mathbf{u}_d \in \mathbf{L}^2(\Omega) \\ \text{and } \theta_d \in L^2(\Omega) \text{ are given.} \end{array}\right. \tag{2.2}$$

3 Well-posedness and Regularity of Solutions for (2.1)

In this section we analyze the existence, uniqueness and regularity of weak solution for system (2.1), which, as was said in Section 1, it is not a simple generalization of the similar ones to deal with Navier-Stokes or related models in fluid mechanics [39].

3.1 Weak Solutions for (2.1)

We introduce the bilinear and trilinear forms $a : \mathbf{X} \times \mathbf{X} \to \mathbb{R}$, $c : \mathbf{X} \times \mathbf{X} \times \mathbf{X} \to \mathbb{R}$, $a_1 : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$, $b_1 : H^1(\Omega) \times \mathbf{X} \to \mathbb{R}$ and $c_1 : \mathbf{X} \times H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$, for the velocity and temperature:

$$a(\mathbf{u}, \mathbf{v}) = \int_\Omega \nabla \mathbf{u} \cdot \nabla \mathbf{v} \, d\Omega, \qquad c(\mathbf{u}, \mathbf{v}, \mathbf{z}) = \int_\Omega [(\mathbf{u} \cdot \nabla)\mathbf{v}] \cdot \mathbf{z} \, d\Omega,$$
$$a_1(\theta, W) = \int_\Omega \nabla\theta \cdot \nabla W \, d\Omega, \qquad b_1(\theta, \mathbf{v}) = \int_\Omega \nabla\theta \cdot \frac{\partial \mathbf{v}}{\partial x_3} \, d\Omega,$$
$$c_1(\mathbf{u}, \theta, W) = \int_\Omega [(\mathbf{u} \cdot \nabla)\theta] W \, d\Omega.$$

Lemma 3.1. *The following relations hold for c and c_1:*

$$c(\mathbf{u}, \mathbf{v}, \mathbf{z}) = -c(\mathbf{u}, \mathbf{z}, \mathbf{v}), \;\; c(\mathbf{u}, \mathbf{v}, \mathbf{v}) = 0, \;\;\; \forall \mathbf{u} \in \mathbf{X}_0, \;\; \forall \mathbf{v}, \mathbf{z} \in \mathbf{H}^1(\Omega), \tag{3.1}$$

$$c_1(\mathbf{u}, \theta, W) = -c_1(\mathbf{u}, W, \theta), \;\; c_1(\mathbf{u}, \theta, \theta) = 0, \;\;\; \forall \mathbf{u} \in \mathbf{X}_0, \;\; \forall \theta, W \in H^1(\Omega). \tag{3.2}$$

Proof: Considering that $\mathbf{u} \in \mathbf{X}_0$, i.e. $\mathbf{u} = 0$ on Γ_0, $u_3 = 0$ on Γ_1 and div $\mathbf{u} = 0$, and the normal vector $\mathbf{n}$ on Γ_1 is $\mathbf{n} = (0, 0, 1)$, we obtain that $\mathbf{u} \cdot \mathbf{n} = 0$ on $\partial\Omega$. Therefore, the proof follows as in Lemma 2.2 in [15], p. 285. $\square$

Lemma 3.2. *([35]) Assume that Ω is a bounded domain with boundary $\partial\Omega$ Lipschitz, and $\partial\Omega = \Gamma_0 \cup \Gamma_1$ with $\Gamma_1 \subseteq \{x_3 = C\}$, C a constant. If $\theta \in H^2(\Omega)$ then*

$$\int_{\Gamma_1} \frac{\partial \theta}{\partial x_1} v_1 + \frac{\partial \theta}{\partial x_2} v_2 \, dS = \int_{\Omega} \nabla\theta \cdot \frac{\partial \mathbf{v}}{\partial x_3} \, d\Omega, \;\; \forall \mathbf{v} \in \mathbf{X}_0.$$

Motivated by the formula of integration by parts and using Lemma 3.2, we obtain the following weak formulation of System (2.1).

Definition 3.3. *A pair $[\mathbf{u}, \theta] \in \mathbf{X} \times H^1(\Omega)$ is said a weak solution of (2.1) if*

$$Pr\, a(\mathbf{u}, \mathbf{v}) + Pr\, M\, b_1(\theta, \mathbf{v}) + c(\mathbf{u}, \mathbf{u}, \mathbf{v}) = \langle f(\theta), \mathbf{v} \rangle\,, \;\; \forall \mathbf{v} \in \mathbf{X}_0, \tag{3.3}$$

$$c_1(\mathbf{u}, \theta, W) + a_1(\theta, W) + \langle B\theta, W \rangle_{\Gamma_1} = \langle \phi_1, W \rangle_{\Gamma_0 \setminus \{x_3 = 0\}}\,, \;\; \forall W \in Y, \tag{3.4}$$

$$\mathbf{u} = \mathbf{g} \;\; on \;\; \Gamma_0^1, \quad \mathbf{u} = \mathbf{u}^0 \;\; on \;\; \Gamma_0^2 \quad and \quad \theta = \phi_2 \;\; on \;\; \{x_3 = 0\}, \tag{3.5}$$

where $\langle f(\theta), \mathbf{v} \rangle = \int_{\Omega} Pr\,(b + R\theta)\, \boldsymbol{e}_3 \cdot \mathbf{v} \, d\Omega.$

3.2 Existence of Weak Solutions

In order to prove the existence of a solution to the problem (3.3)-(3.5) we reduce the problem to an auxiliary problem with homogeneous conditions for the velocity $\mathbf{u}$ on Γ_0 and the temperature θ on $\{x_3 = 0\}$. For that, we will use the Hopf Lemma (see Lemma 4.2 of [18], p. 28). Notice that if $\mathbf{u}^0 \in \widetilde{\mathbf{H}}_{00}^{1/2}(\Gamma_0^2)$ and $\mathbf{g} \in \widetilde{\mathbf{H}}_{00}^{1/2}(\Gamma_0^1)$, then there exist $\widetilde{\mathbf{u}}^0 \in \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$ and $\widetilde{\mathbf{g}} \in \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$ such that

$$\widetilde{\mathbf{u}}^0 \,|_{\Gamma_0^2} = \mathbf{u}^0, \;\; \widetilde{\mathbf{u}}^0 \,|_{\partial\Omega \setminus \Gamma_0^2} = 0, \;\; \int_{\Gamma_0^2} \mathbf{u}^0 \cdot \mathbf{n} = 0, \;\; \mathbf{u}^0 \cdot \mathbf{n} = 0 \;\text{ on } \Gamma_0^2 \setminus \{x_3 = 0\},$$

$$\widetilde{\mathbf{g}} \,|_{\Gamma_0^1} = \mathbf{g}, \;\; \widetilde{\mathbf{g}} \,|_{\partial\Omega \setminus \Gamma_0^1} = 0, \;\; \int_{\Gamma_0^1} \mathbf{g} \cdot \mathbf{n} = 0, \;\; \mathbf{g} \cdot \mathbf{n} = 0 \;\text{ on } \Gamma_0^1 \setminus \{x_3 = 0\}.$$

Thus, the function $\widetilde{\mathbf{u}}^0 + \widetilde{\mathbf{g}} \in \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$ and

$$\int_{\partial\Omega} (\widetilde{\mathbf{u}}^0 + \widetilde{\mathbf{g}}) \cdot \mathbf{n} = \int_{\Gamma_0^1} \widetilde{\mathbf{g}} \cdot \mathbf{n} + \int_{\Gamma_0^2} \widetilde{\mathbf{u}}^0 \cdot \mathbf{n} = \int_{\Gamma_0^1} \mathbf{g} \cdot \mathbf{n} + \int_{\Gamma_0^2} \mathbf{u}^0 \cdot \mathbf{n} = 0.$$

Therefore, by the Hopf Lemma, there exists a function $\mathbf{u}_\varepsilon = (u_{\varepsilon_1}, u_{\varepsilon_2}, u_{\varepsilon_3})$ which satisfies the conditions

$$\mathbf{u}_\varepsilon \in \mathbf{H}^1(\Omega), \qquad \text{div}\, \mathbf{u}_\varepsilon = 0 \;\text{ in } \Omega, \qquad \mathbf{u}_\varepsilon = \widetilde{\mathbf{u}}^0 + \widetilde{\mathbf{g}} \;\text{ on } \partial\Omega,$$

$$\|\mathbf{u}_\varepsilon\|_{H^1(\Omega)} \leq C \left(\|\mathbf{u}^0\|_{H^{\frac{1}{2}}(\Gamma_0^2)} + \|\mathbf{g}\|_{H^{\frac{1}{2}}(\Gamma_0^1)} \right), \qquad |c(\mathbf{v}, \mathbf{u}_\varepsilon, \mathbf{v})| \leq \varepsilon \, \|\mathbf{v}\|^2_{H^1(\Omega)}\,, \;\; \forall \mathbf{v} \in \mathbf{X}_0,$$

where $C = C(n, \Omega)$ and $\varepsilon > 0$ is a real number arbitrarily small. Notice that $\mathbf{u}_\varepsilon \,|_{\Gamma_0^2} = \mathbf{u}^0$ and $\mathbf{u}_\varepsilon \,|_{\Gamma_0^1} = \mathbf{g}$. Moreover, proceeding as in Lemma 3.1, we can easily prove that the following relations hold:

$$c(\mathbf{u}_\varepsilon, \mathbf{u}, \mathbf{v}) = -c(\mathbf{u}_\varepsilon, \mathbf{v}, \mathbf{u}), \;\; c(\mathbf{u}_\varepsilon, \mathbf{u}, \mathbf{u}) = 0, \;\; \forall \mathbf{u}, \mathbf{v} \in \mathbf{X}_0, \tag{3.6}$$

$$c_1(\mathbf{u}_\varepsilon, \theta, W) = -c_1(\mathbf{u}_\varepsilon, W, \theta), \;\; c_1(\mathbf{u}_\varepsilon, \theta, \theta) = 0, \;\; \forall \theta, W \in Y. \tag{3.7}$$

On the other hand, arguing as in [**?**], we can construct a function $\theta_\delta \in H^1(\Omega)$ such that

$$\theta_\delta = \phi_2 \;\text{ on }\; \{x_3 = 0\}, \quad \frac{\partial \theta_\delta}{\partial \mathbf{n}} + B\theta_\delta = 0 \;\text{ on }\; \Gamma_1, \quad \frac{\partial \theta_\delta}{\partial \mathbf{n}} = \phi_1 \;\text{ on }\; \Gamma_0 \setminus \{x_3 = 0\},$$

$$\|\theta_\delta\|_{L^4(\Omega)} \leq \delta, \quad \|\theta_\delta\|_{H^1(\Omega)} \leq C \left(\|\phi_1\|_{H^{\frac{1}{2}}(\Gamma_0 \setminus \{x_3 = 0\})} + \|\phi_2\|_{H^{\frac{1}{2}}(\{x_3 = 0\})} \right). \tag{3.8}$$

Here δ is an arbitrarily small number and the constant C depends on δ.

Rewriting $[\mathbf{u},\theta] \in \mathbf{X} \times H^1(\Omega)$ in the form $\mathbf{u} = \mathbf{u}_\varepsilon + \widehat{\mathbf{u}}$ and $\theta = \theta_\delta + \widehat{\theta}$ with $\widehat{\mathbf{u}} \in \mathbf{X}_0$ and $\widehat{\theta} \in Y$ new unknown functions, from Definition 3.3 we obtain the following nonlinear problem: Find $[\widehat{\mathbf{u}},\widehat{\theta}] \in \mathbf{X}_0 \times Y$ such that

$$\begin{aligned} &Pr\, a(\widehat{\mathbf{u}},\mathbf{v}) + Pr\, M\, b_1(\widehat{\theta},\mathbf{v}) + c(\widehat{\mathbf{u}},\widehat{\mathbf{u}},\mathbf{v}) + c(\widehat{\mathbf{u}},\mathbf{u}_\varepsilon,\mathbf{v}) + c(\mathbf{u}_\varepsilon,\widehat{\mathbf{u}},\mathbf{v}) = \left\langle f(\widehat{\theta}+\theta_\delta),\mathbf{v}\right\rangle \\ &\quad - Pr\, a(\mathbf{u}_\varepsilon,\mathbf{v}) - Pr\, M\, b_1(\theta_\delta,\mathbf{v}) - c(\mathbf{u}_\varepsilon,\mathbf{u}_\varepsilon,\mathbf{v}), \ \ \forall \mathbf{v} \in \mathbf{X}_0, \end{aligned} \tag{3.9}$$

$$\begin{aligned} &c_1(\widehat{\mathbf{u}},\widehat{\theta},W) + c_1(\mathbf{u}_\varepsilon,\widehat{\theta},W) + c_1(\widehat{\mathbf{u}},\theta_\delta,W) + a_1(\widehat{\theta},W) + \left\langle B\widehat{\theta},W\right\rangle_{\Gamma_1} = \langle \phi_1,W\rangle_{\Gamma_0\setminus\{x_3=0\}} \\ &\quad - c_1(\mathbf{u}_\varepsilon,\theta_\delta,W) - a_1(\theta_\delta,W) - \langle B\theta_\delta,W\rangle_{\Gamma_1}, \ \forall W \in Y. \end{aligned} \tag{3.10}$$

Here $\langle f(\theta),\mathbf{v}\rangle$ is as in Definition 3.3.

In order to prove existence of a solution $[\widehat{\mathbf{u}},\widehat{\theta}] \in \mathbf{X}_0 \times Y$ of (3.9)-(3.10), we introduce the mapping $F : \mathbf{X}_0 \to \mathbf{X}_0$ defined by $F(\bar{\mathbf{u}}) = \widehat{\mathbf{u}}$, $\bar{\mathbf{u}} \in \mathbf{X}_0$, such that $[\widehat{\mathbf{u}},\widehat{\theta}] \in \mathbf{X}_0 \times Y$ is the solution of the following linearized problem

$$\begin{aligned} &Pr\, a(\widehat{\mathbf{u}},\mathbf{v}) + Pr\, M\, b_1(\widehat{\theta},\mathbf{v}) + c(\bar{\mathbf{u}},\widehat{\mathbf{u}},\mathbf{v}) + c(\bar{\mathbf{u}},\mathbf{u}_\varepsilon,\mathbf{v}) + c(\mathbf{u}_\varepsilon,\widehat{\mathbf{u}},\mathbf{v}) = \left\langle f(\widehat{\theta}+\theta_\delta),\mathbf{v}\right\rangle \\ &\quad - Pr\, a(\mathbf{u}_\varepsilon,\mathbf{v}) - Pr\, M\, b_1(\theta_\delta,\mathbf{v}) - c(\mathbf{u}_\varepsilon,\mathbf{u}_\varepsilon,\mathbf{v}), \ \ \forall \mathbf{v} \in \mathbf{X}_0, \end{aligned} \tag{3.11}$$

$$\begin{aligned} &c_1(\bar{\mathbf{u}},\widehat{\theta},W) + c_1(\mathbf{u}_\varepsilon,\widehat{\theta},W) + c_1(\bar{\mathbf{u}},\theta_\delta,W) + a_1(\widehat{\theta},W) + \left\langle B\widehat{\theta},W\right\rangle_{\Gamma_1} = \langle \phi_1,W\rangle_{\Gamma_0\setminus\{x_3=0\}} \\ &\quad - c_1(\mathbf{u}_\varepsilon,\theta_\delta,W) - a_1(\theta_\delta,W) - \langle B\theta_\delta,W\rangle_{\Gamma_1}, \ \forall W \in Y. \end{aligned} \tag{3.12}$$

In next lemma, we shall show that the operator $F : \mathbf{X}_0 \to \mathbf{X}_0$ is well-defined.

Lemma 3.4. *Let $\bar{\mathbf{u}} \in \mathbf{X}_0$ and $\phi_1 \in H^{\frac{1}{2}}(\Gamma_0 \setminus \{x_3 = 0\})$. Then there exists a unique weak solution $[\widehat{\mathbf{u}},\widehat{\theta}] \in \mathbf{X}_0 \times Y$ of problem (3.11)-(3.12). Moreover, the following estimates hold*

$$\begin{aligned} \|\widehat{\mathbf{u}}\|_{H^1(\Omega)} \le{}& C\left(|b| + (R+M)\left(\|\widehat{\theta}\|_{H^1(\Omega)} + \|\theta_\delta\|_{H^1(\Omega)}\right) + \frac{1}{Pr}\|\bar{\mathbf{u}}\|_{L^4(\Omega)}\|\mathbf{u}_\varepsilon\|_{L^4(\Omega)}\right) \\ &+C\left(\|\nabla \mathbf{u}_\varepsilon\|_{L^2(\Omega)} + \frac{1}{Pr}\|\mathbf{u}_\varepsilon\|^2_{H^1(\Omega)}\right), \end{aligned} \tag{3.13}$$

$$\|\widehat{\theta}\|_{H^1(\Omega)} \le C\left(\|\phi_1\|_{H^{\frac{1}{2}}(\Gamma_0\setminus\{x_3=0\})} + \|\bar{\mathbf{u}}\|_{L^4(\Omega)}\|\theta_\delta\|_{L^4(\Omega)} + \left(\|\mathbf{u}_\varepsilon\|_{L^4(\Omega)} + 1 + B\right)\|\theta_\delta\|_{H^1(\Omega)}\right), \tag{3.14}$$

where C is a constant independent of $\bar{\mathbf{u}}$, $\widehat{\mathbf{u}}$, ϕ_1 and $\widehat{\theta}$.

Proof: We consider the bilinear continuous mappings $\hat{a} : \mathbf{X}_0 \times \mathbf{X}_0 \to \mathbb{R}$ and $\hat{a}_1 : Y \times Y \to \mathbb{R}$ given by

$$\begin{aligned} &\hat{a}(\widehat{\mathbf{u}},\mathbf{v}) = Pr\, a(\widehat{\mathbf{u}},\mathbf{v}) + c(\bar{\mathbf{u}},\widehat{\mathbf{u}},\mathbf{v}) + c(\mathbf{u}_\varepsilon,\widehat{\mathbf{u}},\mathbf{v}), \ \ \forall \widehat{\mathbf{u}},\mathbf{v} \in \mathbf{X}_0, \\ &\hat{a}_1(\widehat{\theta},W) = c_1(\bar{\mathbf{u}},\widehat{\theta},W) + c_1(\mathbf{u}_\varepsilon,\widehat{\theta},W) + a_1(\widehat{\theta},W) + \left\langle B\widehat{\theta},W\right\rangle_{\Gamma_1}, \ \ \forall \widehat{\theta},W \in Y. \end{aligned}$$

Consequently, we rewrite (3.11) and (3.12) as

$$\hat{a}(\widehat{\mathbf{u}},\mathbf{v}) = \left\langle l_{\widehat{\theta}},\mathbf{v}\right\rangle, \ \forall \mathbf{v} \in \mathbf{X}_0, \tag{3.15}$$

$$\hat{a}_1(\widehat{\theta},W) = \left\langle \widetilde{\phi}_1,W\right\rangle, \ \forall W \in Y, \tag{3.16}$$

where

$$\left\langle l_{\widehat{\theta}},\mathbf{v}\right\rangle = \left\langle f(\widehat{\theta}+\theta_\delta),\mathbf{v}\right\rangle - Pr\, M\, b_1(\widehat{\theta}+\theta_\delta,\mathbf{v}) - c(\bar{\mathbf{u}},\mathbf{u}_\varepsilon,\mathbf{v}) - Pr\, a(\mathbf{u}_\varepsilon,\mathbf{v}) - c(\mathbf{u}_\varepsilon,\mathbf{u}_\varepsilon,\mathbf{v}), \ \forall \mathbf{v} \in \mathbf{X}_0,$$

$$\left\langle \widetilde{\phi}_1, W \right\rangle = \langle \phi_1, W\rangle_{\Gamma_0\setminus\{x_3=0\}} - c_1(\bar{\mathbf{u}}, \theta_\delta, W) - c_1(\mathbf{u}_\varepsilon, \theta_\delta, W) - a_1(\theta_\delta, W) - \langle B\theta_\delta, W\rangle_{\Gamma_1}, \ \forall W \in Y.$$

We can verify that the operator bilinear $\hat{a}_1$ is continuous and coercive on Y and $\widetilde{\phi}_1 \in Y'$. Indeed, the continuity of $\hat{a}_1$ and $\widetilde{\phi}_1$ it follows from the Hölder inequality and Sobolev embeddings. Moreover, the coercivity of $\hat{a}_1$ follows from (3.2), (3.7) and the following generalized Poincaré inequality:

$$\|u\|_{L^2(\Omega)} \leq C\left(\|\nabla u\|_{L^2(\Omega)} + \int_\Sigma |u|\right), \ \forall u \in H^1(\Omega), \tag{3.17}$$

where $C = C(n, \Omega, \Sigma)$ and Σ is an arbitrary portion of $\partial\Omega$ of positive measure (cf. Lemma 10.9 in [30], p. 327; see also [17], p. 56). Therefore, by the Lax-Milgram theorem, there exists a unique $\widehat{\theta} \in Y$ which satisfies equation (3.16). Knowing $\widehat{\theta}$ and inserting it in the equation (3.15), by using the Hölder inequality and Sobolev embeddings we can verify that the operator bilinear $\hat{a}$ is continuous on $\mathbf{X}_0$ and $l_\theta \in \mathbf{X}_0'$. Moreover, from (3.1), (3.6) and using the generalized Poincaré inequality (3.17) we have that $\hat{a}$ is coercive. Therefore, by the Lax-Milgram theorem, there exists a unique $\widehat{\mathbf{u}} \in \mathbf{X}_0$ which satisfies equation (3.15). Finally, setting $\mathbf{v} = \widehat{\mathbf{u}}$ in (3.15), $W = \widehat{\theta}$ in (3.16) and using the generalized Poincaré inequality (3.17), we easily obtain (3.13) and (3.14). $\square$

Now, using the Schauder Fixed Point Theorem, we will prove the existence of a fixed point of F which yields a solution of (3.9)-(3.10). For that, we consider the ball $B_r = \{\widehat{\mathbf{u}} \in \mathbf{X}_0 : \|\widehat{\mathbf{u}}\|_{\mathbf{H}^1(\Omega)} \leq r\} \subseteq \mathbf{X}_0$, where r is a positive constant such that

$$r > C\Bigg(|b| + (R+M)\Big[\|\phi_1\|_{H^{\frac{1}{2}}(\Gamma_0\setminus\{x_3=0\})} + \big(\|\mathbf{u}_\varepsilon\|_{L^4(\Omega)} + 1 + B\big)\|\theta_\delta\|_{H^1(\Omega)}\Big] + \|\nabla\mathbf{u}_\varepsilon\|_{L^2(\Omega)} + \frac{1}{Pr}\|\mathbf{u}_\varepsilon\|^2_{H^1(\Omega)}\Bigg).$$

It follows from (3.13)-(3.14) that $F(B_r) \subseteq B_r$, provided δ be small enough and Pr large enough. Moreover F is completely continuous. This follows from the next inequality

$$\|F(\bar{\mathbf{u}}_1) - F(\bar{\mathbf{u}}_2)\|_{H^1(\Omega)} \leq K\|\bar{\mathbf{u}}_1 - \bar{\mathbf{u}}_2\|_{L^4(\Omega)}, \tag{3.18}$$

and from the compact embedding of $\mathbf{H}^1(\Omega) \hookrightarrow \mathbf{L}^4(\Omega)$, where

$$\begin{aligned} K = &\frac{C}{Pr}\left(\|\widehat{\mathbf{u}}_2\|_{L^4(\Omega)} + \|\mathbf{u}_\varepsilon\|_{L^4(\Omega)}\right) \\ &+ C(M+R)\Big(\|\phi_1\|_{H^{\frac{1}{2}}(\Gamma_0\setminus\{x_3=0\})} + \delta\|\bar{\mathbf{u}}_2\|_{L^4(\Omega)} + (\|\mathbf{u}_\varepsilon\|_{L^4(\Omega)} + 1 + B)\|\theta_\delta\|_{H^1(\Omega)}\Big), \end{aligned}$$

and C is a constant independent of $\bar{\mathbf{u}}_1$ and $\bar{\mathbf{u}}_2$.

Let us prove (3.18). Let $\widehat{\theta}_i \in Y$ be the solution of equation (3.12) corresponding to $\bar{\mathbf{u}}_i \in \mathbf{X}_0$ and set $\widehat{\mathbf{u}}_i = F(\bar{\mathbf{u}}_i)$, for $i = 1, 2$. From (3.11) and (3.12) we obtain

$$\begin{aligned} Pr\, a(\widehat{\mathbf{u}}_1 - \widehat{\mathbf{u}}_2, \mathbf{v}) &+ c(\bar{\mathbf{u}}_1, \widehat{\mathbf{u}}_1 - \widehat{\mathbf{u}}_2, \mathbf{v}) + c(\bar{\mathbf{u}}_1 - \bar{\mathbf{u}}_2, \widehat{\mathbf{u}}_2, \mathbf{v}) = -c(\bar{\mathbf{u}}_1 - \bar{\mathbf{u}}_2, \mathbf{u}_\epsilon, \mathbf{v}) \\ &- c(\mathbf{u}_\epsilon, \widehat{\mathbf{u}}_1 - \widehat{\mathbf{u}}_2, \mathbf{v}) - Pr\, M\, b_1(\widehat{\theta}_1 - \widehat{\theta}_2, \mathbf{v}) + \int_\Omega PrR(\widehat{\theta}_1 - \widehat{\theta}_2)v_3, \ \forall \mathbf{v} \in \mathbf{X}_0, \end{aligned} \tag{3.19}$$

$$\begin{aligned} a_1(\widehat{\theta}_1 - \widehat{\theta}_2, W) &+ \left\langle B(\widehat{\theta}_1 - \widehat{\theta}_2), W\right\rangle_{\Gamma_1} = -c_1(\bar{\mathbf{u}}_1, \widehat{\theta}_1 - \widehat{\theta}_2, W) \\ &- c_1(\bar{\mathbf{u}}_1 - \bar{\mathbf{u}}_2, \widehat{\theta}_2, W) - c_1(\mathbf{u}_\varepsilon, \widehat{\theta}_1 - \widehat{\theta}_2, W) - c_1(\bar{\mathbf{u}}_1 - \bar{\mathbf{u}}_2, \theta_\delta, W), \ \forall W \in Y. \end{aligned} \tag{3.20}$$

Setting $W = \widehat{\theta}_1 - \widehat{\theta}_2$ in (3.20) and using (3.2), (3.7), the Hölder inequality, the continuous embedding $H^1(\Omega) \hookrightarrow L^4(\Omega)$ and the Poincaré inequality, it is not difficult to obtain

$$\|\nabla(\widehat{\theta}_1 - \widehat{\theta}_2)\|_{L^2(\Omega)} \leq C\left(\|\widehat{\theta}_2\|_{H^1(\Omega)} + \|\theta_\delta\|_{H^1(\Omega)}\right)\|\bar{\mathbf{u}}_1 - \bar{\mathbf{u}}_2\|_{L^4(\Omega)}. \tag{3.21}$$

Now, using (3.14) and the Poincaré inequality (3.17), from (3.21) we obtain

$$\|\widehat{\theta}_1 - \widehat{\theta}_2\|_{H^1(\Omega)} \leq C\Big(\|\phi_1\|_{H^{\frac{1}{2}}(\Gamma_0\setminus\{x_3=0\})} + \delta\|\bar{\mathbf{u}}_2\|_{L^4(\Omega)} + (\|\mathbf{u}_\varepsilon\|_{L^4(\Omega)} + 1 + B)\|\theta_\delta\|_{H^1(\Omega)}\Big)\|\bar{\mathbf{u}}_1 - \bar{\mathbf{u}}_2\|_{L^4(\Omega)}. \tag{3.22}$$

Setting $\mathbf{v} = \widehat{\mathbf{u}}_1 - \widehat{\mathbf{u}}_2$ in (3.19) and using (3.1), (3.6), the Hölder inequality, the continuous embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ and the Poincaré inequality, we obtain

$$Pr\,\|\nabla(\widehat{\mathbf{u}}_1 - \widehat{\mathbf{u}}_2)\|_{L^2(\Omega)} \leq (\|\widehat{\mathbf{u}}_2\|_{L^4(\Omega)} + \|\mathbf{u}_\varepsilon\|_{L^4(\Omega)})\|\bar{\mathbf{u}}_1 - \bar{\mathbf{u}}_2\|_{L^4(\Omega)} + C(PrM + PrR)\|\widehat{\theta}_1 - \widehat{\theta}_2\|_{H^1(\Omega)}. \quad (3.23)$$

Then, using the Poincaré inequality, from (3.23) we get

$$\|\widehat{\mathbf{u}}_1 - \widehat{\mathbf{u}}_2\|_{H^1(\Omega)} \leq C\left(\frac{1}{Pr}(\|\widehat{\mathbf{u}}_2\|_{L^4(\Omega)} + \|\mathbf{u}_\varepsilon\|_{L^4(\Omega)})\|\bar{\mathbf{u}}_1 - \bar{\mathbf{u}}_2\|_{L^4(\Omega)} + (M+R)\|\widehat{\theta}_1 - \widehat{\theta}_2\|_{H^1(\Omega)}\right). \quad (3.24)$$

Thus, (3.18) follows from (3.24) and (3.22). Therefore, the Schauder Theorem implies that F has a fixed point $\widehat{\mathbf{u}} = F(\widehat{\mathbf{u}})$. The field $\widehat{\mathbf{u}}$, together with the corresponding function $\widehat{\theta} = \theta_{\widehat{\mathbf{u}}} \in Y$ solving the problem (3.12) for $\bar{\mathbf{u}} = \widehat{\mathbf{u}}$, is a solution to the problem (3.9)-(3.10). We collect this result in the following theorem:

Theorem 3.5. *Let $\phi_1 \in H^{\frac{1}{2}}(\Gamma_0 \setminus \{x_3 = 0\})$, $\phi_2 \in H^{1/2}_{00}(\{x_3 = 0\})$, $\mathbf{u}^0 \in \widetilde{\mathbf{H}}^{1/2}_{00}(\Gamma_0^2)$ and $\mathbf{g} \in \widetilde{\mathbf{H}}^{1/2}_{00}(\Gamma_0^1)$. Then there exists at least one solution $[\mathbf{u}, \theta] \in \widetilde{\mathbf{X}} \times H^1(\Omega)$ of problem (3.3)-(3.5) provided Pr be large enough, and the following estimate holds*

$$\|\mathbf{u}\|_{H^1(\Omega)} + \|\theta\|_{H^1(\Omega)} \leq C\left(\|\mathbf{u}^0\|_{H^{\frac{1}{2}}(\Gamma_0^2)} + \|\mathbf{g}\|_{H^{\frac{1}{2}}(\Gamma_0^1)} + \|\phi_1\|_{H^{\frac{1}{2}}(\Gamma_0 \setminus \{x_3=0\})} + \|\phi_2\|_{H^{\frac{1}{2}}(\{x_3=0\})}\right), \quad (3.25)$$

where the constant C depends linearly of the parameters M, B and R.

3.3 Uniqueness of the Weak Solutions

The purpose of this section is to determine conditions on the boundary data and parameters which guarantee the uniqueness of the weak solution $[\mathbf{u}, \theta] \in \widetilde{\mathbf{X}} \times H^1(\Omega)$ to the problem (3.3)-(3.5). For that, suppose that there exist $[\mathbf{u}_1, \theta_1], [\mathbf{u}_2, \theta_2] \in \widetilde{\mathbf{X}} \times H^1(\Omega)$ weak solutions of system (3.3)-(3.5). Then, defining $\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2$ and $\theta = \theta_1 - \theta_2$, we obtain that $[\mathbf{u}, \theta] \in \mathbf{X}_0 \times Y$ solves the system

$$Pr\, a(\mathbf{u}, \mathbf{v}) + Pr\, M\, b_1(\theta, \mathbf{v}) + c(\mathbf{u}, \mathbf{u}_1, \mathbf{v}) + c(\mathbf{u}_2, \mathbf{u}, \mathbf{v}) = \int_\Omega Pr\ R\ \theta v_3\, d\Omega,\ \forall \mathbf{v} \in \mathbf{X}_0, \quad (3.26)$$

$$c_1(\mathbf{u}, \theta_1, W) + c_1(\mathbf{u}_2, \theta, W) + a_1(\theta, W) + \langle B\theta, W\rangle_{\Gamma_1} = 0,\ \forall W \in Y. \quad (3.27)$$

Proceeding as in Lemma 3.1, we can easily prove that if $\mathbf{u}_2 \in \widetilde{\mathbf{X}}$, $\mathbf{u} \in \mathbf{X}_0$ and $\theta \in Y$, then $c(\mathbf{u}_2, \mathbf{u}, \mathbf{u}) = 0$ and $c_1(\mathbf{u}_2, \theta, \theta) = 0$. Thus, setting $\mathbf{v} = \mathbf{u}$ in (3.26), $W = \theta$ in (3.27), and using the Hölder inequality, Sobolev embeddings and the Poincaré inequality (3.17), we deduce

$$Pr\|\nabla\mathbf{u}\|_{L^2(\Omega)} \leq PrM\|\nabla\theta\|_{L^2(\Omega)} + C\|\nabla\mathbf{u}\|_{L^2(\Omega)}\|\nabla\mathbf{u}_1\|_{L^2(\Omega)} + CPrR\|\nabla\theta\|_{L^2(\Omega)}, \quad (3.28)$$

$$\|\nabla\theta\|_{L^2(\Omega)} \leq C\|\nabla\mathbf{u}\|_{L^2(\Omega)}\|\nabla\theta_1\|_{L^2(\Omega)}. \quad (3.29)$$

Using (3.29) in (3.28), we find

$$Pr\|\nabla\mathbf{u}\|_{L^2(\Omega)} \leq C(\|\nabla\mathbf{u}_1\|_{L^2(\Omega)} + (PrM + PrR)\|\nabla\theta_1\|_{L^2(\Omega)})\|\nabla\mathbf{u}\|_{L^2(\Omega)}.$$

Now, taking into account that $[\mathbf{u}_1, \theta_1]$ and $[\mathbf{u}_2, \theta_2]$ are weak solutions to the problem (3.3)-(3.5), then from Theorem 3.5, we have that $[\mathbf{u}_1, \theta_1]$ and $[\mathbf{u}_2, \theta_2]$ satisfy the estimate (3.25), which imply that

$$Pr\|\nabla\mathbf{u}\|_{L^2(\Omega)} \leq C(Pr(M+R)+1)\Big[\|\mathbf{u}^0\|_{H^{\frac{1}{2}}(\Gamma_0^2)} + \|\mathbf{g}\|_{H^{\frac{1}{2}}(\Gamma_0^1)} + \|\phi_1\|_{H^{\frac{1}{2}}(\Gamma_0\setminus\{x_3=0\})} + \|\phi_2\|_{H^{\frac{1}{2}}(\{x_3=0\})}\Big]\|\nabla\mathbf{u}\|_{L^2(\Omega)},$$

where the constant C depends almost linearly of the parameters M, B y R. Therefore, if the condition

$$Pr - C(Pr(M+R)+1)\left[\|\mathbf{u}^0\|_{H^{\frac{1}{2}}(\Gamma_0^2)} + \|\mathbf{g}\|_{H^{\frac{1}{2}}(\Gamma_0^1)} + \|\phi_1\|_{H^{\frac{1}{2}}(\Gamma_0\setminus\{x_3=0\})} + \|\phi_2\|_{H^{\frac{1}{2}}(\{x_3=0\})}\right] > 0 \quad (3.30)$$

is satisfied, we conclude that $\|\nabla\mathbf{u}\|_{L^2(\Omega)} = 0$, and consequently $\mathbf{u} = 0$, which implies that $\mathbf{u}_1 = \mathbf{u}_2$. Moreover, using this fact in (3.29), we obtain that $\|\nabla\theta\|_{L^2(\Omega)} = 0$, and consequently $\theta = 0$, which implies that $\theta_1 = \theta_2$. Thus we have proved the following theorem:

Theorem 3.6. *Let $\phi_1 \in H^{\frac{1}{2}}(\Gamma_0 \setminus \{x_3 = 0\})$, $\phi_2 \in H^{1/2}_{00}(\{x_3 = 0\})$ $\mathbf{u}^0 \in \widetilde{\mathbf{H}}^{1/2}_{00}(\Gamma^2_0)$ and $\mathbf{g} \in \widetilde{\mathbf{H}}^{1/2}_{00}(\Gamma^1_0)$. If the condition (3.30) is satisfied, then the problem (3.3)-(3.5) has a unique solution $[\mathbf{u}, \theta] \in \widetilde{\mathbf{X}} \times H^1(\Omega)$. Moreover, the solution $[\mathbf{u}, \theta]$ satisfies the estimate (3.25).*

Remark 3.7. *Observe that the condition*

$$Pr - C(Pr(M + R) + 1)\left[\|\mathbf{u}^0\|_{H^{\frac{1}{2}}(\Gamma^2_0)} + \|\mathbf{g}\|_{H^{\frac{1}{2}}(\Gamma^1_0)} + \|\phi_1\|_{H^{\frac{1}{2}}(\Gamma_0 \setminus \{x_3=0\})} + \|\phi_2\|_{H^{\frac{1}{2}}(\{x_3=0\})}\right] > 0$$

is verified if either the functions $\mathbf{u}^0$, $\mathbf{g}$, ϕ_1 and ϕ_2 are small, or if the coefficients M, R and B are small. In particular, for small values of M, R and B and boundary data $\mathbf{u}^0 = 0$, $\mathbf{g} = 0$, $\phi_1 = 0$ and $\phi_2 = \theta_c$, the basic solution $[\mathbf{u}_b, \theta_b, p_b]$ given by (1.4) is unique.

3.4 Regularity

In Subsection 3.2 was demonstrated the existence of a weak solution $[\mathbf{u}, \theta] \in \widetilde{\mathbf{X}} \times H^1(\Omega)$ to the problem (3.3)-(3.5); however, taking into account the tangential and normal derivatives of the temperature at the boundary, we need to prove that $\theta \in H^2(\Omega)$ (see Lemma 3.2). In this subsection we analyze the following regularity problem for the weak solution $\theta \in H^1(\Omega)$: Given $\mathbf{u} \in \widetilde{\mathbf{X}}$, find $\theta \in H^2(\Omega)$ such that

$$\begin{cases} -\Delta\theta = -(\mathbf{u} \cdot \nabla)\theta & \text{in } \Omega, \\ \frac{\partial\theta}{\partial\mathbf{n}} + B\theta = 0 & \text{on } \Gamma_1, \\ \frac{\partial\theta}{\partial\mathbf{n}} = \phi_1 & \text{on } \Gamma_2, \\ \theta = \phi_2 & \text{on } \Gamma_3, \end{cases} \tag{3.31}$$

where $\Gamma_1 := \{x_3 = 1\}$, $\Gamma_3 := \{x_3 = 0\}$ and $\Gamma_2 := \partial\Omega \setminus \{\Gamma_1 \cup \Gamma_3\}$ (see Figure 1).

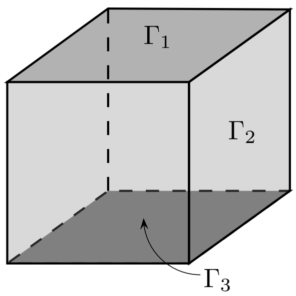

Figure 1: Representation of $\partial\Omega$.

In this subsection. we will use the following space

$$H^{3/2}_{00}(\Gamma) = \{v \in L^2(\Gamma) : \text{ there exists } g \in H^{\frac{3}{2}}(\partial\Omega),\ g|_\Gamma = v,\ g|_{\partial\Omega\setminus\Gamma} = 0\}.$$

Theorem 3.8. *Let $\phi_1 \in H^{\frac{1}{2}}(\Gamma_2)$, $\phi_2 \in H^{3/2}_{00}(\Gamma_3)$ and $f \in L^p(\Omega)$ with $\frac{6}{5} < p \leq 2$. Then, the system*

$$\begin{cases} -\Delta\theta = f & \textit{in } \Omega, \\ \frac{\partial\theta}{\partial\mathbf{n}} + B\theta = 0 & \textit{on } \Gamma_1, \\ \frac{\partial\theta}{\partial\mathbf{n}} = \phi_1 & \textit{on } \Gamma_2, \\ \theta = \phi_2 & \textit{on } \Gamma_3, \end{cases} \tag{3.32}$$

has a solution $\theta \in W^{2,p}(\Omega)$.

Proof: We first convert the problem (3.32) with Robin, Neumann and Dirichlet conditions, in a boundary problem with only Dirichlet and Neumann conditions. For that, we will adapt the ideas of [23], Section 2. First, we consider the functions $\eta(x_3)$ and $\tilde{\theta}$ defined by:

$$\eta(x_3) = \exp[B(2x_3 - \frac{1}{2}x_3^2)] \qquad \text{and} \qquad \tilde{\theta} = \eta\theta.$$

Since B is constant, it is easy to check that problem (3.32) is equivalent to find $\tilde{\theta} \in W^{2,p}(\Omega)$, such that

$$\begin{cases} -\Delta\tilde{\theta} = \tilde{f} & \text{in } \Omega, \\ \frac{\partial\tilde{\theta}}{\partial\mathbf{n}} = 0 & \text{on } \Gamma_1, \\ \frac{\partial\tilde{\theta}}{\partial\mathbf{n}} = \tilde{\phi}_1 & \text{on } \Gamma_2, \\ \tilde{\theta} = \phi_2 & \text{on } \Gamma_3, \end{cases} \tag{3.33}$$

where $\tilde{f} = -\theta\eta'' - 2\frac{\partial\theta}{\partial x_3}\eta' + \eta f$ and $\tilde{\phi}_1 = \eta\phi_1$. Taking into account that $\phi_2 \in H_{00}^{3/2}(\Gamma_3)$, we consider the function $\widetilde{\phi}_2 \in H^{\frac{3}{2}}(\partial\Omega)$ such that $\widetilde{\phi}_2 = \phi_2$ on Γ_3 and $\widetilde{\phi}_2 = 0$ on $\partial\Omega\backslash\Gamma_3$. By the lifting Theorem, we have that exists $\Phi_2 \in H^2(\Omega)$ such that $\Phi_2|_{\partial\Omega} = \widetilde{\phi}_2$. Considering $\hat{\theta} = \tilde{\theta} - \Phi_2$, it is not difficult to verify that problem (3.33) is equivalent to find $\hat{\theta} \in W^{2,p}(\Omega)$, such that

$$\begin{cases} -\Delta\hat{\theta} = \hat{f} & \text{in } \Omega, \\ \frac{\partial\hat{\theta}}{\partial\mathbf{n}} = \phi_3 & \text{on } \Gamma_1, \\ \frac{\partial\hat{\theta}}{\partial\mathbf{n}} = \hat{\phi}_1 & \text{on } \Gamma_2, \\ \hat{\theta} = 0 & \text{on } \Gamma_3, \end{cases} \tag{3.34}$$

where $\hat{f} = \tilde{f} + \Delta\Phi_2$, $\phi_3 = -\frac{\partial\Phi_2}{\partial\mathbf{n}}\big|_{\Gamma_1}$ and $\hat{\phi}_1 = \tilde{\phi}_1 - \frac{\partial\Phi_2}{\partial\mathbf{n}}\big|_{\Gamma_2}$. In order to find the solution $\hat{\theta}$ of problem (3.34), we decompose $\hat{\theta}$ as the sum $\hat{\theta} = \theta_1 + \theta_2 + \theta_3$, where θ_1, θ_2 and θ_3 solve respectively the following problems:

$$\begin{cases} -\Delta\theta_1 = \hat{f} & \text{in } \Omega, \\ \frac{\partial\theta_1}{\partial\mathbf{n}} = 0 & \text{on } \Gamma_1, \\ \frac{\partial\theta_1}{\partial\mathbf{n}} = 0 & \text{on } \Gamma_2, \\ \theta_1 = 0 & \text{on } \Gamma_3, \end{cases} \tag{3.35}$$

$$\begin{cases} -\Delta\theta_2 = 0 & \text{in } \Omega, \\ \frac{\partial\theta_2}{\partial\mathbf{n}} = 0 & \text{on } \Gamma_1, \\ \frac{\partial\theta_2}{\partial\mathbf{n}} = \hat{\phi}_1 & \text{on } \Gamma_2, \\ \theta_2 = 0 & \text{on } \Gamma_3, \end{cases} \tag{3.36}$$

$$\begin{cases} -\Delta\theta_3 = 0 & \text{in } \Omega, \\ \frac{\partial\theta_3}{\partial\mathbf{n}} = \phi_3 & \text{on } \Gamma_1, \\ \frac{\partial\theta_3}{\partial\mathbf{n}} = 0 & \text{on } \Gamma_2, \\ \theta_3 = 0 & \text{on } \Gamma_3. \end{cases} \tag{3.37}$$

In order to prove the existence of $\theta_1, \theta_2, \theta_3 \in W^{2,p}(\Omega)$, we require the following preliminary result whose proof follows from Theorem 1 in [11] (see also [12]).

Theorem 3.9. *If $F \in L^q(\Omega)$ with $\frac{6}{5} < q < \infty$, then the weak solution to the problem*

$$\begin{cases} -\Delta\omega = F & \text{in } \Omega, \\ \frac{\partial \omega}{\partial \mathbf{n}} = 0 & \text{on } \Gamma_1 \cup \Gamma_2, \\ \omega = 0 & \text{on } \Gamma_3, \end{cases}$$

belongs to the space $W^{2,q}(\Omega)$.

Thus, by Theorem 3.9, if $\hat{f} \in L^p(\Omega)$ with $\frac{6}{5} < p \leq 2$, then the system (3.35) has solution $\theta_1 \in W^{2,p}(\Omega)$. We remember that $\hat{f} = -\theta\eta'' - 2\frac{\partial\theta}{\partial x_3}\eta' + \eta f + \Delta\Phi_2$. Observe that as $\theta \in H^1(\Omega) \hookrightarrow L^2(\Omega)$ then $\frac{\partial\theta}{\partial x_3} \in L^2(\Omega)$. Moreover since $\eta(x_3) = \exp[B(2x_3 - \frac{1}{2}x_3^2)]$, then $\eta' = B(2-x_3)\eta$ and $\eta'' = -B\eta + B^2(2-x_3)^2\eta$. Recalling that $0 \leq x_3 \leq 1$, we deduce that η, η' y η'' belong to $L^2(\Omega)$. Finally, as $\Phi_2 \in H^2(\Omega)$, $\Delta\Phi_2 \in L^2(\Omega)$, and as by initial hypothesis $f \in L^p(\Omega)$ with $\frac{6}{5} < p \leq 2$, we conclude that $\hat{f} \in L^p(\Omega)$ with $\frac{6}{5} < p \leq 2$. Thus, the system (3.35) has solution $\theta_1 \in W^{2,p}(\Omega)$.

On the other hand, observe that for finding $\theta_2, \theta_3 \in W^{2,p}(\Omega)$ solutions of (3.36) and (3.37) respectively, we can not use directly Theorem 3.9, because these systems have nonhomogeneous boundary conditions. Therefore, for solving the problem (3.36), we first divide Γ_2 in four parts Γ_2^i with $i = 1,2,3,4$, as showed in Figure 2, and then we decompose the solution θ_2 as the sum $\theta_2 = \theta_2^1 + \theta_2^2 + \theta_2^3 + \theta_2^4$, where θ_2^i $(i = 1,2,3,4)$ solve:

$$\begin{cases} -\Delta\theta_2^i = 0 & \text{in } \Omega, \\ \frac{\partial \theta_2^i}{\partial \mathbf{n}} = \hat{\phi}_1^i & \text{on } \Gamma_2^i, \\ \frac{\partial \theta_2^i}{\partial \mathbf{n}} = 0 & \text{on } \Gamma_1 \cup (\Gamma_2 \setminus \Gamma_2^i), \\ \theta_2^i = 0 & \text{on } \Gamma_3, \end{cases} \tag{3.38}$$

where $\hat{\phi}_1^i$ is defined by $\hat{\phi}_1^i = \hat{\phi}_1\Big|_{\Gamma_2^i}$ on Γ_2^i, and $\hat{\phi}_1^i = 0$ on $\Gamma_2 \setminus \Gamma_2^i$, $i = 1,2,3,4$.

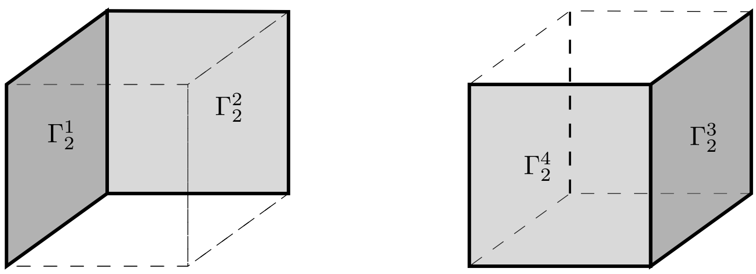

Figure 2: Division of Γ_2.

For solving problems (3.38), we will adapt the ideas of [23], Section 2. In the case $i = 1$, we divide the boundary of Γ_2^1, denoted by $\partial\Gamma_2^1$, as follows: $\partial\Gamma_2^1 = \Gamma_2^{11} \cup \Gamma_2^{12} \cup \Gamma_2^{13} \cup \Gamma_2^{14}$ (see Figure 3), and we construct a function ψ_1 as a solution of the heat equation:

$$\begin{cases} \frac{\partial \psi_1}{\partial x_2} = \Delta\psi_1 & \text{in } \Gamma_2^1 \times (0,\infty), \\ \frac{\partial \psi_1}{\partial \mathbf{n}} = 0 & \text{on } \Gamma_2^{1i} \times (0,\infty), \quad i = 1,2,4, \\ \psi_1 = 0 & \text{on } \Gamma_2^{13} \times (0,\infty), \\ \psi_1(x_1,0,x_3) = \hat{\phi}_1(x_1,0,x_3) & \text{on } \Gamma_2^1. \end{cases} \tag{3.39}$$

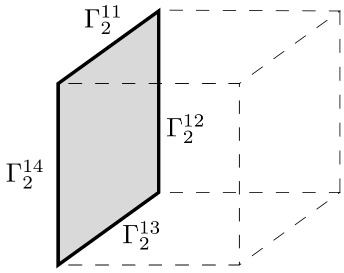

Figure 3: Division of $\partial\Gamma_2^1$.

By standard methods (cf. [30], Ch. 10) we can verify that there exists a solution $\psi_1 \in H^2(\Omega)$ for problem (3.39). Moreover, considering the following function

$$T_1(x_1,x_2,x_3) = (1-x_2^2)\int_{x_2}^{L}\psi_1(x_1,z,x_3)\,dz - x_2^2\frac{(L^2-1)}{2L}\psi_1(x_1,L,x_3), \quad (x_1,x_2,x_3)\in\Gamma_2^1\times(0,\infty),$$

we can easily see that T_1 satisfies the boundary conditions in (3.38) (for $i=1$). Moreover, taking into account that $\psi_1 \in H^2(\Omega)$, we deduce that $T_1 \in H^2(\Omega)$. Additionally, as $T_1 \in H^2(\Omega)$ then $-\Delta T_1 \in L^2(\Omega)$, and consequently $-\Delta T_1 \in L^p(\Omega)$ for $\frac{6}{5} < p \le 2$. Thus, by Theorem 3.9, the solution $\tilde{T}_1$ of the system

$$\begin{cases} \Delta\tilde{T}_1 = -\Delta T_1 & \text{in } \Omega, \\ \frac{\partial \tilde{T}_1}{\partial \mathbf{n}} = 0 & \text{on } \Gamma_1\cup\Gamma_2, \\ \tilde{T}_1 = 0 & \text{on } \Gamma_3, \end{cases}$$

belongs to $W^{2,p}(\Omega)$. In conclusion, considering $\theta_2^1 = T_1 + \tilde{T}_1$, we obtain that $\theta_2^1 \in W^{2,p}(\Omega)$ satisfies the system (3.38) for $i=1$. Analogously, we can find solutions θ_2^2, θ_2^3, θ_2^4 and θ_3 in $W^{2,p}(\Omega)$, for problems (3.38) (for $i=2,3,4$) and (3.37) respectively. Thus, considering $\theta_2 = \theta_2^1+\theta_2^2+\theta_2^3+\theta_2^4$ we deduce that $\theta_2 \in W^{2,p}(\Omega)$ is a solution to the system (3.36). Therefore, it was verified the existence of $\theta_1,\theta_2,\theta_3 \in W^{2,p}(\Omega)$ solutions of (3.35), (3.36) and (3.37) respectively, and the theorem is proven. □

Now, taking into account Theorem 3.8, we prove the following theorem which guarantees the existence of solution of problem (3.31).

Theorem 3.10. *Let $\phi_1 \in H^{\frac{1}{2}}(\Gamma_2)$, $\phi_2 \in H_{00}^{3/2}(\Gamma_3)$, $\mathbf{u} \in \widetilde{\mathbf{X}}$ and $\theta \in H^1(\Omega)$ weak solution of system (3.31). Then, the solution θ belongs to the space $H^2(\Omega)$.*

Proof: First, observe that as $\mathbf{u} \in \widetilde{\mathbf{X}} \subset \mathbf{H}^1(\Omega)$ then using Sobolev embeddings we obtain that $\mathbf{u} \in \mathbf{L}^6(\Omega)$. Moreover, as $\theta \in H^1(\Omega)$, $\nabla\theta \in L^2(\Omega)$ and consequently $-(\mathbf{u}\cdot\nabla)\theta \in L^{\frac{3}{2}}(\Omega)$. Thus, by Theorem 3.8 we conclude that the problem (3.31) has solution $\theta \in W^{2,\frac{3}{2}}(\Omega)$. Analogously, since $\theta \in W^{2,\frac{3}{2}}(\Omega)$, $\nabla\theta \in W^{1,\frac{3}{2}}(\Omega)$, and consequently, using the Sobolev embedding $W^{1,\frac{3}{2}}(\Omega) \hookrightarrow L^3(\Omega)$ we deduce that $\nabla\theta \in L^3(\Omega)$. Therefore, $-(\mathbf{u}\cdot\nabla)\theta \in L^2(\Omega)$ and from Theorem 3.8, we conclude that the solution θ of problem (3.31) belongs to $H^2(\Omega)$. □

Remark 3.11. *Taking into account that the geometry of Ω corresponds with a cube, we are able to obtain $\mathbf{H}^2$-regularity for the velocity $\mathbf{u}$. For that, we can apply the results of L^p-regularity for the Stokes problem in polyhedral domains (see [10, 24, 33, 34]).*

4 Existence of Optimal Solutions

In this section we will prove the existence of an optimal solution for Problem (2.2). We define the set of admissible solutions of Problem (2.2) as follows:

$$\mathcal{S}_{ad} := \{\mathbf{z} \equiv [\mathbf{u}, \theta, \mathbf{g}, \phi_1, \phi_2] \in \widetilde{\mathbf{X}} \times H^1(\Omega) \times \mathbf{U}_1 \times \mathcal{U}_2 \times \mathcal{U}_3 \text{ such that}$$
$$\mathcal{J}(\mathbf{z}) < \infty \text{ and the equations (3.3)-(3.5) hold}\}.$$

Then, we have the following result:

Theorem 4.1. *Under the conditions of Theorem 3.5, if one of the conditions* (i) *or* (ii) *given in (2.2) is satisfied, then the problem (2.2) has at least one solution, that is, there exists at least a* $\hat{\mathbf{z}} \equiv [\hat{\mathbf{u}}, \hat{\theta}, \hat{\mathbf{g}}, \hat{\phi}_1, \hat{\phi}_2] \in \mathcal{S}_{ad}$ *such that*

$$\mathcal{J}(\hat{\mathbf{z}}) = \min_{\mathbf{z} \in \mathcal{S}_{ad}} \mathcal{J}(\mathbf{z}).$$

Proof: From Theorem 3.5 we have that $\mathcal{S}_{ad}$ is nonempty. Denote by $(\mathbf{z}_m) = ([\mathbf{u}_m, \theta_m, \mathbf{g}_m, \phi_{1_m}, \phi_{2_m}]) \subset \mathcal{S}_{ad}$, $m \in \mathbb{N}$, a minimizing sequence for which $\lim_{m\to\infty} \mathcal{J}(\mathbf{z}_m) = \min_{\mathbf{z}\in \mathcal{S}_{ad}} \mathcal{J}(\mathbf{z})$. If one of the conditions (i) or (ii) is satisfied, then there exist constants C_1, C_2 and C_3, independent of m, such that $\|\mathbf{g}_m\|_{H^{\frac{1}{2}}(\Gamma_0^1)} \leq C_1$, $\|\phi_{1_m}\|_{H^{\frac{1}{2}}(\Gamma_0 \setminus \{x_3=0\})} \leq C_2$ and $\|\phi_{2_m}\|_{H^{\frac{1}{2}}(\{x_3=0\})} \leq C_3$. Thus, from Theorem 3.5 we conclude that there exist constants C_4 and C_5, independent of m, such that $\|\mathbf{u}_m\|_{H^1(\Omega)} \leq C_4$ and $\|\theta_m\|_{H^1(\Omega)} \leq C_5$. Therefore, since $\mathbf{U}_1$, $\mathcal{U}_2$ and $\mathcal{U}_3$ are closed convex subsets of $\widetilde{\mathbf{H}}_{00}^{1/2}(\Gamma_0^1)$, $H^{\frac{1}{2}}(\Gamma_0 \setminus \{x_3 = 0\})$ and $H_{00}^{1/2}(\{x_3 = 0\})$ respectively, we obtain $\hat{\mathbf{z}} \equiv [\hat{\mathbf{u}}, \hat{\theta}, \hat{\mathbf{g}}, \hat{\phi}_1, \hat{\phi}_2] \in \mathbf{H}^1(\Omega) \times H^1(\Omega) \times \mathbf{U}_1 \times \mathcal{U}_2 \times \mathcal{U}_3$ such that, for some subsequence of $(\mathbf{z}_m)_{m\in\mathbb{N}} \subset \mathcal{S}_{ad}$ still denoted by $(\mathbf{z}_m)_{m\in\mathbb{N}}$, we have

$$\begin{aligned}
&\mathbf{u}_m \rightharpoonup \hat{\mathbf{u}} \text{ in } \mathbf{H}^1(\Omega) \text{ and strongly in } \mathbf{L}^p(\Omega),\ 2 \leq p < 6, \\
&\theta_m \rightharpoonup \hat{\theta} \text{ in } H^1(\Omega) \text{ and strongly in } L^l(\Omega),\ 2 \leq l < 6, \\
&\mathbf{g}_m \rightharpoonup \hat{\mathbf{g}} \text{ in } \mathbf{H}^{\frac{1}{2}}(\Gamma_0^1) \text{ and strongly in } \mathbf{L}^2(\Gamma_0^1), \\
&\phi_{1_m} \rightharpoonup \hat{\phi}_1 \text{ in } H^{\frac{1}{2}}(\Gamma_0 \setminus \{x_3 = 0\}) \text{ and strongly in } L^2(\Gamma_0 \setminus \{x_3 = 0\}), \\
&\phi_{2_m} \rightharpoonup \hat{\phi}_2 \text{ in } H^{\frac{1}{2}}(\{x_3 = 0\}) \text{ and strongly in } L^2(\{x_3 = 0\}).
\end{aligned} \tag{4.1}$$

Since $\mathbf{u}_m|_{\Gamma_0^1} = \mathbf{g}_m$, $\mathbf{u}_m|_{\Gamma_0^2} = \mathbf{u}^0$ and $\theta_m|_{\{x_3=0\}} = \phi_{2_m}$, it follows from the properties of the trace operators that $\hat{\mathbf{u}}|_{\Gamma_0^1} = \hat{\mathbf{g}}$, $\hat{\mathbf{u}}|_{\Gamma_0^2} = \mathbf{u}^0$ and $\hat{\theta}\,|_{\{x_3=0\}} = \hat{\phi}_2$; so, $\hat{\mathbf{z}}$ satisfies the boundary conditions (3.5). Moreover, since the third component of $\mathbf{u}_m$ denoted by u_{m_3} is equal to 0 on Γ_1 for all $m \in \mathbb{N}$, then from the continuity of the trace operator we obtain $\hat{u}_3 = 0$ on Γ_1. Also, using (4.1) we obtain that div $\mathbf{u}_m \rightharpoonup$ div $\hat{\mathbf{u}}$ in $\mathbf{L}^2(\Omega)$, and given that div $\mathbf{u}_m = 0$ for all $m \in \mathbb{N}$, we conclude that div $\hat{\mathbf{u}} = 0$. Moreover, as $\hat{\mathbf{u}} = \hat{\mathbf{g}}$ on Γ_0^1 and $\hat{\mathbf{u}} = \mathbf{u}^0$ on Γ_0^2, we obtain that $\hat{\mathbf{u}} \cdot \mathbf{n} = 0$ on $\Gamma_0 \setminus \{x_3 = 0\}$. Therefore, we conclude that $\hat{\mathbf{u}} \in \widetilde{\mathbf{X}}$. A standard procedure permits to pass the limit, as m goes to ∞, in the variational formulation (3.3)-(3.4), and we obtain that $\hat{\mathbf{z}}$ satisfies the weak formulation (3.3)-(3.5). Consequently we have that $\hat{\mathbf{z}} \equiv [\hat{\mathbf{u}}, \hat{\theta}, \hat{\mathbf{g}}, \hat{\phi}_1, \hat{\phi}_2] \in \mathcal{S}_{ad}$, and then

$$\mathcal{J}(\hat{\mathbf{z}}) \geq \inf_{\mathbf{z}\in\mathcal{S}_{ad}} \mathcal{J}(\mathbf{z}).$$

Finally, recalling that the functional $\mathcal{J}$ is weakly lower semicontinuous on $\mathcal{S}_{ad}$, we have that

$$\mathcal{J}(\hat{\mathbf{z}}) = \min_{\mathbf{z}\in\mathcal{S}_{ad}} \mathcal{J}(\mathbf{z}).$$

□

Remark 4.2. *Let* $[\mathbf{u}_b, \theta_b]$ *the basic solution to the problem (1.2)-(1.3) given by (1.4). From Theorem 4.1, we can obtain the existence of controls* $[\mathbf{g}, \phi_1, \phi_2] \in \mathbf{U}_1 \times \mathcal{U}_2 \times \mathcal{U}_3$ *and a weak solution* $[\mathbf{u}, \theta] \in \widetilde{\mathbf{X}} \times H^1(\Omega)$ *of the problem (3.3)-(3.5), such that the functional (2.2) is minimized if we consider* $\mathbf{u}_d = \mathbf{u}_b$ *and* $\theta_d = \theta_b$*, the basic state.*

5 Necessary Optimality Conditions and an Optimality System

In order to obtain first-order optimality conditions, we start by considering the following Banach spaces: $\mathbb{G} = \widetilde{\mathbf{X}} \times H^1(\Omega) \times \mathbf{U}_1 \times \mathcal{U}_2 \times \mathcal{U}_3$ and $\mathbb{H} = \mathbf{X}_0 \times Y$, with the usual inner products and norms. Moreover, if Γ is a connected subset of the boundary $\partial\Omega$, we define the trace spaces

$$H_e^{1/2}(\Gamma) = \{v \in L^2(\Gamma) :\ \text{there exists } g \in H^{\frac{1}{2}}(\partial\Omega),\ g|_\Gamma = v\},$$

$$\widetilde{\mathbf{H}}_e^{1/2}(\Gamma) = \left\{ \mathbf{v} \in \mathbf{L}^2(\Gamma) : \exists\, \mathbf{g} \in \mathbf{H}^{\frac{1}{2}}(\partial\Omega),\ \mathbf{g}|_\Gamma = \mathbf{v}, \int_\Gamma \mathbf{g}\cdot\mathbf{n} = 0,\ \mathbf{g}\cdot\mathbf{n} = 0 \text{ on } \Gamma \setminus \{x_3 = 0\},\ g_3 = 0 \text{ on } \Gamma_1 \right\},$$

which are closed subspaces of $H^{1/2}(\Gamma)$ and $\mathbf{H}^{1/2}(\Gamma)$, respectively. Also, let $\mathbf{u}_{\mathbf{g}}^0$ defined by

$$\mathbf{u}_{\mathbf{g}}^0 = \begin{cases} \mathbf{g} & \text{on } \Gamma_0^1, \\ \mathbf{u}^0 & \text{on } \Gamma_0^2. \end{cases}$$

Then, taking into account that $\mathbf{g} \in \widetilde{\mathbf{H}}_{00}^{1/2}(\Gamma_0^1)$ and $\mathbf{u}^0 \in \widetilde{\mathbf{H}}_{00}^{1/2}(\Gamma_0^2)$, we can easily prove that $\mathbf{u}_{\mathbf{g}}^0 \in \widetilde{\mathbf{H}}_{00}^{1/2}(\Gamma_0)$. Also, we consider the following operators $\mathcal{F}_1 : \mathbb{G} \to \mathbf{X}_0'$, $\mathcal{F}_2 : \mathbb{G} \to Y'$, $\mathcal{F}_3 : \mathbb{G} \to \widetilde{\mathbf{H}}_e^{1/2}(\Gamma_0)$ and $\mathcal{F}_4 : \mathbb{G} \to H_e^{1/2}(\{x_3 = 0\})$, defined at each point $\mathbf{z} := [\mathbf{u}, \theta, \mathbf{g}, \phi_1, \phi_2]$ by:

$$\begin{cases} \langle \mathcal{F}_1(\mathbf{z}), \mathbf{v}\rangle &=& Pr\, a(\mathbf{u},\mathbf{v}) + Pr\, M\, b_1(\theta,\mathbf{v}) + c(\mathbf{u},\mathbf{u},\mathbf{v}) - \langle f(\theta), \mathbf{v}\rangle,\ \forall \mathbf{v} \in \mathbf{X}_0, \\ \langle \mathcal{F}_2(\mathbf{z}), W\rangle &=& c_1(\mathbf{u},\theta,W) + a_1(\theta,W) + \langle B\theta, W\rangle_{\Gamma_1} - \langle \phi_1, W\rangle_{\Gamma_0\setminus\{x_3=0\}},\ \forall W \in Y, \\ \mathcal{F}_3(\mathbf{z}) &=& \mathbf{u}|_{\Gamma_0} - \mathbf{u}_{\mathbf{g}}^0, \\ \mathcal{F}_4(\mathbf{z}) &=& \theta|_{\{x_3=0\}} - \phi_2. \end{cases}$$

In order to simplify the notation, let us denote by $\mathbb{M}$ the space

$$\mathbb{M} \equiv \mathbf{X}_0' \times Y' \times \widetilde{\mathbf{H}}_e^{1/2}(\Gamma_0) \times H_e^{1/2}(\{x_3 = 0\}),$$

and define the operator

$$\boldsymbol{F} : \mathbb{G} \to \mathbb{M}, \ \text{ such that } \boldsymbol{F}(\mathbf{z}) := [\mathcal{F}_1(\mathbf{z}), \mathcal{F}_2(\mathbf{z}), \mathcal{F}_3(\mathbf{z}), \mathcal{F}_4(\mathbf{z})].$$

Then the optimal control problem (2.2) is equivalent to:

$$\begin{cases} \text{Find } \mathbf{z} := [\mathbf{u}, \theta, \mathbf{g}, \phi_1, \phi_2] \in \mathbb{G} \text{ such that the functional} \\[2ex] \mathcal{J}[\mathbf{u},\theta,\mathbf{g},\phi_1,\phi_2] = \frac{\gamma_1}{2}\|\text{rot } \mathbf{u}\|^2_{L^2(\Omega)} + \frac{\gamma_2}{2}\|\mathbf{u}-\mathbf{u}_d\|^2_{L^2(\Omega)} + \frac{\gamma_3}{2}\|\theta-\theta_d\|^2_{L^2(\Omega)} + \frac{\gamma_4}{2}\|\mathbf{g}\|^2_{H^{\frac{1}{2}}(\Gamma_0^1)} \\ \qquad\qquad + \frac{\gamma_5}{2}\|\phi_1\|^2_{H^{\frac{1}{2}}(\Gamma_0\setminus\{x_3=0\})} + \frac{\gamma_6}{2}\|\phi_2\|^2_{H^{\frac{1}{2}}(\{x_3=0\})}, \\[2ex] \text{is minimized subject to } \langle \boldsymbol{F}(\mathbf{z}), [\mathbf{v}, W]\rangle = [\langle \mathcal{F}_1(\mathbf{z}),\mathbf{v}\rangle, \langle \mathcal{F}_2(\mathbf{z}), W\rangle, \mathcal{F}_3(\mathbf{z}), \mathcal{F}_4(\mathbf{z})] = [\mathbf{0}, 0, \mathbf{0}, 0]. \end{cases} \tag{5.1}$$

5.1 Existence of Lagrange Multipliers

In this subsection, we will prove the existence of Lagrange multipliers. For that, first we will establish a regularity condition for an optimal solution $\hat{\mathbf{z}} \equiv [\hat{\mathbf{u}}, \hat{\theta}, \hat{\mathbf{g}}, \hat{\phi}_1, \hat{\phi}_2] \in S_{ad}$, as was established in [40], p. 50. We follows the ideas of [14]. We start by establishing the following two lemmas related to the Fréchet differentiability of $\boldsymbol{F}$ and $\mathcal{J}$.

Lemma 5.1. *The operator $\boldsymbol{F}$ is Fréchet differentiable with respect to $\mathbf{z} = [\mathbf{u}, \theta, \mathbf{g}, \phi_1, \phi_2] \in \mathbb{G}$. Moreover, at an arbitrary point $\hat{\mathbf{z}} = [\hat{\mathbf{u}}, \hat{\theta}, \hat{\mathbf{g}}, \hat{\phi}_1, \hat{\phi}_2] \in \mathbb{G}$, the Fréchet derivative operator of $\boldsymbol{F}$ with respect to $\mathbf{z}$ is the linear and bounded operator $\boldsymbol{F}_{\mathbf{z}}(\hat{\mathbf{z}}) : \mathbb{G} \to \mathbb{M}$ such that at each point $\mathbf{t} = [\mathbf{h}_1, h_2, \mathbf{r}, \varrho, \tau] \in \mathbb{G}$, is defined by:*

$$\begin{cases} \langle \mathcal{F}_{1\mathbf{z}}(\hat{\mathbf{z}})\mathbf{t}, \mathbf{v}\rangle = Pr\, a(\mathbf{h}_1,\mathbf{v}) + Pr\, M\, b_1(h_2,\mathbf{v}) + c(\hat{\mathbf{u}},\mathbf{h}_1,\mathbf{v}) + c(\mathbf{h}_1,\hat{\mathbf{u}},\mathbf{v}) - PrR(h_2, v_3)_{L^2(\Omega)}, \\ \langle \mathcal{F}_{2\mathbf{z}}(\hat{\mathbf{z}})\mathbf{t}, W\rangle = c_1(\hat{\mathbf{u}}, h_2, W) + c_1(\mathbf{h}_1, \hat{\theta}, W) + a_1(h_2, W) + \langle B h_2, W\rangle_{\Gamma_1} - \langle \varrho, W\rangle_{\Gamma_0\setminus\{x_3=0\}}, \\ \mathcal{F}_{3\mathbf{z}}(\hat{\mathbf{z}})\mathbf{t} = \mathbf{h}_1|_{\Gamma_0} - \mathcal{B}_1\mathbf{r}, \\ \mathcal{F}_{4\mathbf{z}}(\hat{\mathbf{z}})\mathbf{t} = h_2|_{\{x_3=0\}} - \tau, \end{cases} \tag{5.2}$$

for all $[\mathbf{v}, W] \in \mathbb{H}$, *where* $\mathcal{B}_1 \in \mathcal{L}(\widetilde{\mathbf{H}}_{00}^{1/2}(\Gamma_0^1), \widetilde{\mathbf{H}}_{00}^{1/2}(\Gamma_0))$ *is defined by*

$$\mathcal{B}_1\mathbf{r} := \begin{cases} \mathbf{r} & on \quad \Gamma_0^1, \\ \mathbf{0} & on \quad \Gamma_0^2. \end{cases} \tag{5.3}$$

Lemma 5.2. *The functional $\mathcal{J}$ is Fréchet differentiable with respect to $\mathbf{z} = [\mathbf{u}, \theta, \mathbf{g}, \phi_1, \phi_2] \in \mathbb{G}$. Moreover, at an arbitrary point $\hat{\mathbf{z}} = [\hat{\mathbf{u}}, \hat{\theta}, \hat{\mathbf{g}}, \hat{\phi}_1, \hat{\phi}_2] \in \mathbb{G}$, the Fréchet derivative operator of $\mathcal{J}$ with respect to $\mathbf{z}$ is the linear and bounded operator $\mathcal{J}_{\mathbf{z}}(\hat{\mathbf{z}}) : \mathbb{G} \to \mathbb{R}$ such that at each point $\mathbf{t} = [\mathbf{h}_1, h_2, \mathbf{r}, \varrho, \tau] \in \mathbb{G}$, is defined by:*

$$\begin{aligned} \mathcal{J}_{\mathbf{z}}(\hat{\mathbf{z}})\mathbf{t} = {} & \gamma_1(rot\ \hat{\mathbf{u}}, rot\ \mathbf{h}_1)_{L^2(\Omega)} + \gamma_2(\hat{\mathbf{u}} - \mathbf{u}_d, \mathbf{h}_1)_{L^2(\Omega)} + \gamma_3(\hat{\theta} - \theta_d, h_2)_{L^2(\Omega)} + \gamma_4(\hat{\mathbf{g}}, \mathbf{r})_{H^{\frac{1}{2}}(\Gamma_0^1)} \\ & + \gamma_5(\hat{\phi}_1, \varrho)_{H^{\frac{1}{2}}(\Gamma_0 \setminus \{x_3=0\})} + \gamma_6(\hat{\phi}_2, \tau)_{H^{\frac{1}{2}}(\{x_3=0\})}. \end{aligned} \tag{5.4}$$

In the next lemma, we will give a condition to assure that $\hat{\mathbf{z}} \in \mathcal{S}_{ad}$ satisfies the regular point condition (see [40], p. 50). Thereafter the existence of Lagrange multipliers is shown.

Lemma 5.3. *Let $\hat{\mathbf{z}} \equiv [\hat{\mathbf{u}}, \hat{\theta}, \hat{\mathbf{g}}, \hat{\phi}_1, \hat{\phi}_2] \in \mathcal{S}_{ad}$ be a feasible solution for the problem (5.1). If Pr is large enough and M, R are small enough such that*

$$\beta_0 := \min\left\{Pr - C\left(Pr(M+R) + \|\hat{\mathbf{u}}\|_{H^1(\Omega)} + \|\hat{\theta}\|^2_{H^1(\Omega)}\right), \frac{1}{2} - CPr(R+M)\right\} > 0, \tag{5.5}$$

where C is some positive constant, which only depends on the domain Ω, then $\hat{\mathbf{z}}$ satisfies the regular point condition.

Proof: Given $[\mathbf{a}, b, \mathbf{c}, d] \in \mathbb{M}$, it is sufficient to show the existence of $\mathbf{t} = [\mathbf{h}_1, h_2, \mathbf{r}, \varrho, \tau] \in \mathbb{G}$ such that

$$Pr\, a(\mathbf{h}_1, \mathbf{v}) + Pr\, M\, b_1(h_2, \mathbf{v}) + c(\hat{\mathbf{u}}, \mathbf{h}_1, \mathbf{v}) + c(\mathbf{h}_1, \hat{\mathbf{u}}, \mathbf{v}) - PrR(h_2, v_3)_{L^2(\Omega)} = \langle \mathbf{a}, \mathbf{v}\rangle,\ \forall \mathbf{v} \in \mathbf{X}_0, \tag{5.6}$$

$$c_1(\hat{\mathbf{u}}, h_2, W) + c_1(\mathbf{h}_1, \hat{\theta}, W) + a_1(h_2, W) + \langle Bh_2, W\rangle_{\Gamma_1} - \langle \varrho, W\rangle_{\Gamma_0\setminus\{x_3=0\}} = \langle b, W\rangle,\ \forall W \in Y, \tag{5.7}$$

$$\mathbf{h}_1|_{\Gamma_0} = \mathbf{c} + \mathcal{B}_1(\mathbf{r} - \hat{\mathbf{g}}), \tag{5.8}$$

$$h_2|_{\{x_3=0\}} = d + (\tau - \hat{\phi}_2). \tag{5.9}$$

Setting $[\mathbf{r}, \varrho, \tau] = [\hat{\mathbf{g}}, \hat{\phi}_1, \hat{\phi}_2]$, we have that $\mathbf{h}_1|_{\Gamma_0} = \mathbf{c}$ and $h_2|_{\{x_3=0\}} = d$. Then, proceeding as in the beginning of Subsection 3.2, we can prove that there exist $[\mathbf{h}_1^\epsilon, h_2^\delta] \in \widetilde{\mathbf{X}} \times H^1(\Omega)$ such that $\mathbf{h}_1^\epsilon|_{\Gamma_0} = \mathbf{c}$ and $h_2^\delta|_{\{x_3=0\}} = d$. Therefore, rewriting the unknowns $\mathbf{h}_1, h_2$ in the form $\mathbf{h}_1 = \mathbf{h}_1^\epsilon + \widetilde{\mathbf{h}}_1$, $h_2 = h_2^\delta + \widetilde{h}_2$ with $[\widetilde{\mathbf{h}}_1, \widetilde{h}_2] \in \mathbb{H}$ new unknown functions, from (5.6)-(5.9), we obtain the following linear system:

$$Pr\, a(\widetilde{\mathbf{h}}_1, \mathbf{v}) + Pr\, M\, b_1(\widetilde{h}_2, \mathbf{v}) + c(\hat{\mathbf{u}}, \widetilde{\mathbf{h}}_1, \mathbf{v}) + c(\widetilde{\mathbf{h}}_1, \hat{\mathbf{u}}, \mathbf{v}) - PrR(\widetilde{h}_2, v_3)_{L^2(\Omega)} = \langle \widetilde{\mathbf{a}}, \mathbf{v}\rangle,\ \forall \mathbf{v} \in \mathbf{X}_0, \tag{5.10}$$

$$c_1(\hat{\mathbf{u}}, \widetilde{h}_2, W) + c_1(\widetilde{\mathbf{h}}_1, \hat{\theta}, W) + a_1(\widetilde{h}_2, W) + \left\langle B\widetilde{h}_2, W\right\rangle_{\Gamma_1} = \langle \widetilde{b}, W\rangle,\ \forall W \in Y, \tag{5.11}$$

where

$$\langle \widetilde{\mathbf{a}}, \mathbf{v}\rangle = \langle \mathbf{a}, \mathbf{v}\rangle - Pr\, a(\mathbf{h}_1^\epsilon, \mathbf{v}) - Pr\, M\, b_1(h_2^\delta, \mathbf{v}) - c(\hat{\mathbf{u}}, \mathbf{h}_1^\epsilon, \mathbf{v}) - c(\mathbf{h}_1^\epsilon, \hat{\mathbf{u}}, \mathbf{v}) + PrR(h_2^\delta, v_3)_{L^2(\Omega)},$$

$$\langle \widetilde{b}, W\rangle = \langle b, W\rangle - c_1(\hat{\mathbf{u}}, h_2^\delta, W) - c_1(\mathbf{h}_1^\epsilon, \hat{\theta}, W) - a_1(h_2^\delta, W) - \langle Bh_2^\delta, W\rangle_{\Gamma_1} + \left\langle \hat{\phi}_1, W\right\rangle_{\Gamma_0\setminus\{x_3=0\}}.$$

In order to prove the existence of a solution for (5.10)-(5.11), we will apply the Lax-Milgram theorem. For that, we consider the bilinear form $A : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ defined by

$$\begin{aligned} A([\widetilde{\mathbf{h}}_1, \widetilde{h}_2], [\mathbf{v}, W]) = {} & Pr\, a(\widetilde{\mathbf{h}}_1, \mathbf{v}) + Pr\, M\, b_1(\widetilde{h}_2, \mathbf{v}) + c(\hat{\mathbf{u}}, \widetilde{\mathbf{h}}_1, \mathbf{v}) + c(\widetilde{\mathbf{h}}_1, \hat{\mathbf{u}}, \mathbf{v}) - PrR(\widetilde{h}_2, v_3)_{L^2(\Omega)} \\ & + c_1(\hat{\mathbf{u}}, \widetilde{h}_2, W) + c_1(\widetilde{\mathbf{h}}_1, \hat{\theta}, W) + a_1(\widetilde{h}_2, W) + \left\langle B\widetilde{h}_2, W\right\rangle_{\Gamma_1}, \end{aligned} \tag{5.12}$$

and $I : \mathbb{H} \to \mathbb{R}$ defined by $I[\mathbf{v}, W] := \langle \widetilde{\mathbf{a}}, \mathbf{v}\rangle + \langle \widetilde{b}, W\rangle$. Thus, we rewrite (5.10)-(5.11) as

$$A([\widetilde{\mathbf{h}}_1, \widetilde{h}_2], [\mathbf{v}, W]) = I[\mathbf{v}, W]. \tag{5.13}$$

It is not difficult to prove that $A(\cdot,\cdot)$ is continuous and $I \in \mathbb{H}'$. Now we prove the $\mathbb{H}\times\mathbb{H}$-coercivity of A. For that, taking $[\mathbf{v}, W] = [\widetilde{\mathbf{h}}_1, \widetilde{h}_2]$ in (5.12), and using the Hölder, Poincaré and Young inequalities and Sobolev embeddings we get

$$\begin{aligned}
A([\widetilde{\mathbf{h}}_1, \widetilde{h}_2], [\widetilde{\mathbf{h}}_1, \widetilde{h}_2]) &= Pr\|\nabla\widetilde{\mathbf{h}}_1\|^2_{L^2(\Omega)} + Pr\, M\, b_1(\widetilde{h}_2, \widetilde{\mathbf{h}}_1) + c(\widetilde{\mathbf{h}}_1, \hat{\mathbf{u}}, \widetilde{\mathbf{h}}_1) - PrR(\widetilde{h}_2, \widetilde{h}_{1_3})_{L^2(\Omega)}\\
&\quad + c_1(\widetilde{\mathbf{h}}_1, \hat{\theta}, \widetilde{h}_2) + \|\nabla\widetilde{h}_2\|^2_{L^2(\Omega)} + B\|\widetilde{h}_2\|^2_{L^2(\Gamma_1)}\\
&\geq \left(Pr - CPr(M+R) - C\|\hat{\mathbf{u}}\|_{H^1(\Omega)} - C\|\hat{\theta}\|^2_{H^1(\Omega)}\right)\|\nabla\widetilde{\mathbf{h}}_1\|^2_{L^2(\Omega)}\\
&\quad + \left(1 - CPr(R+M) - \frac{1}{2}\right)\|\nabla\widetilde{h}_2\|^2_{L^2(\Omega)}\\
&\geq \beta_0\|[\widetilde{\mathbf{h}}_1, \widetilde{h}_2]\|^2_{\mathbb{H}},
\end{aligned} \tag{5.14}$$

where $\beta_0 = C\min\left\{Pr - C\left(Pr(M+R) + \|\hat{\mathbf{u}}\|_{H^1(\Omega)} + \|\hat{\theta}\|^2_{H^1(\Omega)}\right), \frac{1}{2} - CPr(R+M)\right\} > 0$. Therefore, from (5.13) and (5.14) and the Lax-Milgram theorem we conclude the existence of $[\widetilde{\mathbf{h}}_1, \widetilde{h}_2] \in \mathbb{H}$ solution of (5.10)-(5.11), and consequently, we obtain that $[\mathbf{h}_1, h_2] \in \widetilde{\mathbf{X}} \times H^1(\Omega)$ is solution of (5.6)-(5.9). □

In the next theorem, we will prove the existence of Lagrange multipliers provided a local optimal solution $\hat{\mathbf{z}} \equiv [\hat{\mathbf{u}}, \hat{\theta}, \hat{\mathbf{g}}, \hat{\phi}_1, \hat{\phi}_2] \in S_{ad}$ verifies the regular point condition (see Lemma 5.3).

Theorem 5.4. *Let $\hat{\mathbf{z}} \equiv [\hat{\mathbf{u}}, \hat{\theta}, \hat{\mathbf{g}}, \hat{\phi}_1, \hat{\phi}_2] \in \mathcal{S}_{ad}$ be a local optimal solution for the control problem (5.1) and assume (5.5). Then, there exist Lagrange multipliers $[\boldsymbol{\lambda}_1, \lambda_2, \boldsymbol{\lambda}_3, \lambda_4] \in \mathbb{H}\times(\widetilde{\mathbf{H}}_e^{1/2}(\Gamma_0))'\times(H_e^{1/2}(\{x_3=0\}))'$ such that for all $[\mathbf{h}_1, h_2, \mathbf{r}, \varrho, \tau] \in \widetilde{\mathbf{X}} \times H^1(\Omega) \times \mathcal{C}(\hat{\mathbf{g}}) \times \mathcal{C}(\hat{\phi}_1) \times \mathcal{C}(\hat{\phi}_2)$ it holds:*

$$\begin{aligned}
&\gamma_1(rot\ \hat{\mathbf{u}}, rot\ \mathbf{h}_1)_{L^2(\Omega)} + \gamma_2(\hat{\mathbf{u}} - \mathbf{u}_d, \mathbf{h}_1)_{L^2(\Omega)} + \gamma_3(\hat{\theta} - \theta_d, h_2)_{L^2(\Omega)} + \gamma_5(\hat{\phi}_1, \varrho)_{H^{\frac{1}{2}}(\Gamma_0\setminus\{x_3=0\})}\\
&+\gamma_4(\hat{\mathbf{g}}, \mathbf{r})_{H^{\frac{1}{2}}(\Gamma_0^1)} + \gamma_6(\hat{\phi}_2, \tau)_{H^{\frac{1}{2}}(\{x_3=0\})} - Pr\, a(\mathbf{h}_1, \boldsymbol{\lambda}_1) - Pr\, M\, b_1(h_2, \boldsymbol{\lambda}_1) - c(\hat{\mathbf{u}}, \mathbf{h}_1, \boldsymbol{\lambda}_1) - c(\mathbf{h}_1, \hat{\mathbf{u}}, \boldsymbol{\lambda}_1)\\
&+PrR(h_2, \lambda_{1_3})_{L^2(\Omega)} - c_1(\hat{\mathbf{u}}, h_2, \lambda_2) - c_1(\mathbf{h}_1, \hat{\theta}, \lambda_2) - a_1(h_2, \lambda_2) - \langle Bh_2, \lambda_2\rangle_{\Gamma_1} + \langle \varrho, \lambda_2\rangle_{\Gamma_0\setminus\{x_3=0\}}\\
&-\langle \boldsymbol{\lambda}_3, \mathbf{h}_1|_{\Gamma_0} - \mathcal{B}_1\mathbf{r}\rangle_{(\widetilde{\mathbf{H}}_e^{1/2}(\Gamma_0))', \widetilde{\mathbf{H}}_e^{1/2}(\Gamma_0)} - \langle \lambda_4, h_2|_{\{x_3=0\}} - \tau\rangle_{(H_e^{1/2}(\{x_3=0\}))', H_e^{1/2}(\{x_3=0\})} \geq 0,
\end{aligned} \tag{5.15}$$

where $\mathcal{C}(\hat{\mathbf{g}})\times\mathcal{C}(\hat{\phi}_1)\times\mathcal{C}(\hat{\phi}_2) = \Big\{[\omega_1(\mathbf{g} - \hat{\mathbf{g}}), \omega_2(\phi_1 - \hat{\phi}_1), \omega_3(\phi_2 - \hat{\phi}_2)],\ \omega_1, \omega_2, \omega_3 \geq 0,\ [\mathbf{g}, \phi_1, \phi_2] \in \mathbf{U}_1 \times \mathcal{U}_2 \times \mathcal{U}_3\Big\}$.

Proof: From Lemma 5.3, $\hat{\mathbf{z}} \in \mathcal{S}_{ad}$ satisfies the regular point condition. Then, there exist Lagrange multipliers $[\boldsymbol{\lambda}_1, \lambda_2, \boldsymbol{\lambda}_3, \lambda_4] \in \mathbb{H} \times (\widetilde{\mathbf{H}}_e^{1/2}(\Gamma_0))' \times (H_e^{1/2}(\{x_3=0\}))'$ such that

$$\begin{aligned}
&\mathcal{J}_{\mathbf{z}}(\hat{\mathbf{z}})\mathbf{h} - \langle \mathcal{F}_{1\mathbf{z}}(\hat{\mathbf{z}})\mathbf{h}, \boldsymbol{\lambda}_1\rangle_{\mathbf{X}_0', \mathbf{X}_0} - \langle \mathcal{F}_{2\mathbf{z}}(\hat{\mathbf{z}})\mathbf{h}, \lambda_2\rangle_{Y',Y}\\
&\qquad - \langle \boldsymbol{\lambda}_3, \mathcal{F}_{3\mathbf{z}}(\hat{\mathbf{z}})\mathbf{h}\rangle_{(\widetilde{\mathbf{H}}_e^{1/2}(\Gamma_0))', \widetilde{\mathbf{H}}_e^{1/2}(\Gamma_0)} - \langle \lambda_4, \mathcal{F}_{4\mathbf{x}}(\hat{\mathbf{z}})\mathbf{h}\rangle_{(H_e^{1/2}(\{x_3=0\}))', H_e^{1/2}(\{x_3=0\})} \geq 0,
\end{aligned}$$

for all $[\mathbf{h}_1, h_2, \mathbf{r}, \varrho, \tau] \in \widetilde{\mathbf{X}} \times H^1(\Omega) \times \mathcal{C}(\hat{\mathbf{g}}) \times \mathcal{C}(\hat{\phi}_1) \times \mathcal{C}(\hat{\phi}_2)$. Thus, the proof of theorem follows from (5.2)-(5.4). □

5.2 Optimality System

In this subsection, we derive the equations that are satisfied by the Lagrange multipliers $\boldsymbol{\eta} = [\boldsymbol{\lambda}_1, \lambda_2, \boldsymbol{\lambda}_3, \lambda_4]$ provided by Theorem 5.4.

Theorem 5.5. *(Adjoint Equations) Let $\hat{\mathbf{z}} \equiv [\hat{\mathbf{u}}, \hat{\theta}, \hat{\mathbf{g}}, \hat{\phi}_1, \hat{\phi}_2] \in \mathcal{S}_{ad}$ be an optimal solution for the control problem (5.1) and assume (5.5). Then, there exist functions (Lagrange multipliers) $\boldsymbol{\eta} = [\boldsymbol{\lambda}_1, \lambda_2, \boldsymbol{\lambda}_3, \lambda_4] \in \mathbb{H} \times (\widetilde{\mathbf{H}}_e^{1/2}(\Gamma_0))' \times (H_e^{1/2}(\{x_3 = 0\}))'$ which satisfy, in a variational sense, the following adjoint equations to the control problem (5.1):*

$$\left\{\begin{array}{l} Pr\Delta\boldsymbol{\lambda}_1 + (\hat{\mathbf{u}}\cdot\nabla)\boldsymbol{\lambda}_1 - \nabla^T\hat{\mathbf{u}}\cdot\boldsymbol{\lambda}_1 - \lambda_2\nabla\hat{\theta} = \gamma_1 rot(rot\ \hat{\mathbf{u}}) - \gamma_2(\hat{\mathbf{u}}-\mathbf{u}_d)\ \ in\ \Omega, \\ \Delta\lambda_2 + (\hat{\mathbf{u}}\cdot\nabla)\lambda_2 + PrM\,div\left(\dfrac{\partial\boldsymbol{\lambda}_1}{\partial x_3}\right) + PrR\ \lambda_{1_3} = -\gamma_3(\hat{\theta}-\theta_d)\ \ in\ \Omega, \\ div\,\boldsymbol{\lambda}_1 = 0\ \ in\ \Omega, \\ \boldsymbol{\lambda}_1 = 0\ \ on\ \Gamma_0, \qquad \lambda_{1_3} = 0\ \ on\ \Gamma_1, \qquad \boldsymbol{\lambda}_3 = \gamma_1(rot\ \hat{\mathbf{u}}\times\mathbf{n}) - Pr(\nabla\boldsymbol{\lambda}_1\cdot\mathbf{n})\ \ on\ \Gamma_0, \\ \gamma_1(rot\ \hat{\mathbf{u}}\times\mathbf{n}) - Pr(\nabla\boldsymbol{\lambda}_1\cdot\mathbf{n}) = 0\ \ on\ \Gamma_1, \qquad B\lambda_2 + \left(\nabla\lambda_2 + PrM\dfrac{\partial\boldsymbol{\lambda}_1}{\partial x_3}\right)\cdot\mathbf{n} = 0\ \ on\ \Gamma_1, \\ \lambda_4 = -\left(\nabla\lambda_2 + PrM\dfrac{\partial\boldsymbol{\lambda}_1}{\partial x_3}\right)\cdot\mathbf{n}\ \ on\ \{x_3=0\}, \qquad \lambda_2 = 0\ \ on\ \{x_3 = 0\}, \\ \left(\nabla\lambda_2 + PrM\dfrac{\partial\boldsymbol{\lambda}_1}{\partial x_3}\right)\cdot\mathbf{n} = 0\ \ on\ \Gamma_0\setminus\{x_3=0\}. \end{array}\right. \tag{5.16}$$

Proof: From (5.15) we obtain, taking $[\mathbf{r}, \varrho, \tau] = [\mathbf{0}, 0, 0]$, that for all $[\mathbf{h}_1, h_2] \in \widetilde{\mathbf{X}} \times H^1(\Omega)$,

$$\begin{aligned} &\gamma_1(\mathrm{rot}\ \hat{\mathbf{u}}, \mathrm{rot}\ \mathbf{h}_1)_{L^2(\Omega)} + \gamma_2(\hat{\mathbf{u}}-\mathbf{u}_d, \mathbf{h}_1)_{L^2(\Omega)} + \gamma_3(\hat{\theta}-\theta_d, h_2)_{L^2(\Omega)} - Pr\,a(\mathbf{h}_1, \boldsymbol{\lambda}_1) - PrM\,b_1(h_2, \boldsymbol{\lambda}_1) \\ &\quad -c(\hat{\mathbf{u}}, \mathbf{h}_1, \boldsymbol{\lambda}_1) - c(\mathbf{h}_1, \hat{\mathbf{u}}, \boldsymbol{\lambda}_1) + PrR(h_2, \lambda_{1_3})_{L^2(\Omega)} - c_1(\hat{\mathbf{u}}, h_2, \lambda_2) - c_1(\mathbf{h}_1, \hat{\theta}, \lambda_2) - a_1(h_2, \lambda_2) \\ &- \langle Bh_2, \lambda_2\rangle_{\Gamma_1} - \langle \boldsymbol{\lambda}_3, \mathbf{h}_1|_{\Gamma_0}\rangle_{(\widetilde{\mathbf{H}}_e^{1/2}(\Gamma_0))', \widetilde{\mathbf{H}}_e^{1/2}(\Gamma_0)} - \langle \lambda_4, h_2|_{\{x_3=0\}}\rangle_{(H_e^{1/2}(\{x_3=0\}))', H_e^{1/2}(\{x_3=0\})} = 0. \end{aligned} \tag{5.17}$$

Taking $h_2 = 0$ in (5.17), we get

$$\begin{aligned} &\gamma_1(\mathrm{rot}\ \hat{\mathbf{u}}, \mathrm{rot}\ \mathbf{h}_1)_{L^2(\Omega)} + \gamma_2(\hat{\mathbf{u}}-\mathbf{u}_d, \mathbf{h}_1)_{L^2(\Omega)} - Pr\,a(\mathbf{h}_1, \boldsymbol{\lambda}_1) - c(\hat{\mathbf{u}}, \mathbf{h}_1, \boldsymbol{\lambda}_1) \\ &-c(\mathbf{h}_1, \hat{\mathbf{u}}, \boldsymbol{\lambda}_1) - c_1(\mathbf{h}_1, \hat{\theta}, \lambda_2) - \langle \boldsymbol{\lambda}_3, \mathbf{h}_1|_{\Gamma_0}\rangle_{(\widetilde{\mathbf{H}}_e^{1/2}(\Gamma_0))', \widetilde{\mathbf{H}}_e^{1/2}(\Gamma_0)} = 0, \quad \forall \mathbf{h}_1 \in \widetilde{\mathbf{X}}, \end{aligned} \tag{5.18}$$

and thus, using the Green formula, we obtain

$$\begin{aligned} &-\gamma_1\int_\Omega \mathrm{rot}(\mathrm{rot}\ \hat{\mathbf{u}})\cdot\mathbf{h}_1\,d\Omega + \gamma_1\int_{\partial\Omega}(\mathrm{rot}\ \hat{\mathbf{u}}\times\mathbf{n})\cdot\mathbf{h}_1\,dS + \gamma_2\int_\Omega(\hat{\mathbf{u}}-\mathbf{u}_d)\cdot\mathbf{h}_1\,d\Omega + Pr\int_\Omega \Delta\boldsymbol{\lambda}_1\cdot\mathbf{h}_1\,d\Omega \\ &\quad -Pr\int_{\partial\Omega}\frac{\partial\lambda_1}{\partial\mathbf{n}}\cdot\mathbf{h}_1\,dS + \int_\Omega[(\hat{\mathbf{u}}\cdot\nabla)\boldsymbol{\lambda}_1]\cdot\mathbf{h}_1\,d\Omega - \int_\Omega\nabla^T\hat{\mathbf{u}}\cdot\boldsymbol{\lambda}_1\cdot\mathbf{h}_1\,d\Omega - \int_\Omega\lambda_2\nabla\hat{\theta}\cdot\mathbf{h}_1\,d\Omega \\ &\quad -\langle \boldsymbol{\lambda}_3, \mathbf{h}_1|_{\Gamma_0}\rangle_{(\widetilde{\mathbf{H}}_e^{1/2}(\Gamma_0))', \widetilde{\mathbf{H}}_e^{1/2}(\Gamma_0)} = 0, \quad \forall \mathbf{h}_1 \in \widetilde{\mathbf{X}}. \end{aligned} \tag{5.19}$$

Similarly, taking $\mathbf{h}_1 = 0$ in (5.17), we get

$$\begin{aligned} &\gamma_3(\hat{\theta}-\theta_d, h_2)_{L^2(\Omega)} - PrM\ b_1(h_2, \boldsymbol{\lambda}_1) + PrR\ (h_2, \lambda_{1_3})_{L^2(\Omega)} - c_1(\hat{\mathbf{u}}, h_2, \lambda_2) \\ &-a_1(h_2, \lambda_2) - \langle Bh_2, \lambda_2\rangle_{\Gamma_1} - \langle \lambda_4, h_2|_{\{x_3=0\}}\rangle_{(H_e^{1/2}(\{x_3=0\}))', H_e^{1/2}(\{x_3=0\})} = 0, \quad \forall h_2 \in H^1(\Omega), \end{aligned} \tag{5.20}$$

and thus, using the Green formula, we obtain

$$\begin{aligned} &\gamma_3\int_\Omega(\hat{\theta}-\theta_d)h_2\,d\Omega + PrM\int_\Omega \mathrm{div}\left(\frac{\partial\boldsymbol{\lambda}_1}{\partial x_3}\right)h_2\,d\Omega - PrM\int_{\partial\Omega}\left(\frac{\partial\boldsymbol{\lambda}_1}{\partial x_3}\cdot\mathbf{n}\right)h_2\,dS + PrR\int_\Omega \lambda_{1_3}h_2\,d\Omega \\ &\quad + \int_\Omega[(\hat{\mathbf{u}}\cdot\nabla)\lambda_2]h_2\,d\Omega + \int_\Omega\Delta\lambda_2\,h_2\,d\Omega - \int_{\partial\Omega}\frac{\partial\lambda_2}{\partial\mathbf{n}}\,h_2\,dS - B\int_{\Gamma_1}\lambda_2 h_2\,dS \\ &\quad -\langle \lambda_4, h_2|_{\{x_3=0\}}\rangle_{(H_e^{1/2}(\{x_3=0\}))', H_e^{1/2}(\{x_3=0\})} = 0, \quad \forall h_2 \in H^1(\Omega). \end{aligned} \tag{5.21}$$

Observe that, if additionally we take the test functions $\mathbf{h}_1 \in \mathbf{V}$ in (5.18) and $h_2 \in H_0^1(\Omega)$ in (5.20), we get

$$\begin{aligned} &-\gamma_1\int_\Omega \mathrm{rot}(\mathrm{rot}\ \hat{\mathbf{u}})\cdot\mathbf{h}_1\,d\Omega + \gamma_2\int_\Omega(\hat{\mathbf{u}}-\mathbf{u}_d)\cdot\mathbf{h}_1\,d\Omega + Pr\int_\Omega\Delta\boldsymbol{\lambda}_1\cdot\mathbf{h}_1\,d\Omega \\ &+\int_\Omega[(\hat{\mathbf{u}}\cdot\nabla)\boldsymbol{\lambda}_1]\cdot\mathbf{h}_1\,d\Omega - \int_\Omega\nabla^T\hat{\mathbf{u}}\cdot\boldsymbol{\lambda}_1\cdot\mathbf{h}_1\,d\Omega - \int_\Omega\lambda_2\nabla\hat{\theta}\cdot\mathbf{h}_1\,d\Omega = 0, \quad \forall \mathbf{h}_1 \in \mathbf{V}, \end{aligned} \tag{5.22}$$

$$\gamma_3\int_\Omega(\hat{\theta}-\theta_d)h_2\,d\Omega+PrM\int_\Omega \operatorname{div}\left(\frac{\partial\boldsymbol{\lambda}_1}{\partial x_3}\right)h_2\,d\Omega+PrR\int_\Omega \lambda_{1_3}h_2\,d\Omega$$
$$+\int_\Omega[(\hat{\mathbf{u}}\cdot\nabla)\lambda_2]h_2\,d\Omega+\int_\Omega \Delta\lambda_2\,h_2\,d\Omega=0,\quad \forall h_2\in H_0^1(\Omega), \tag{5.23}$$

and therefore, since $[\mathbf{h}_1,h_2]\in\widetilde{\mathbf{X}}\times H^1(\Omega)$ is arbitrary, from (5.19), (5.21), (5.22) and (5.23), we deduce that $[\boldsymbol{\lambda}_1,\lambda_2,\boldsymbol{\lambda}_3,\lambda_4]$ satisfy, in a variational sense, the adjoint equations (5.16). □

Finally, taking $[\mathbf{h}_1,h_2]=[\mathbf{0},0]$ in (5.15), we obtain

$$\gamma_4(\hat{\mathbf{g}},\mathbf{r})_{H^{\frac12}(\Gamma_0^1)}+\gamma_5(\hat{\phi}_1,\varrho)_{H^{\frac12}(\Gamma_0\setminus\{x_3=0\})}+\gamma_6(\hat{\phi}_2,\tau)_{H^{\frac12}(\{x_3=0\})}+\langle\varrho,\lambda_2\rangle_{\Gamma_0\setminus\{x_3=0\}}$$
$$+\langle\boldsymbol{\lambda}_3,\mathcal{B}_1\mathbf{r}\rangle_{(\widetilde{\mathbf{H}}_e^{1/2}(\Gamma_0))',\widetilde{\mathbf{H}}_e^{1/2}(\Gamma_0)}+\langle\lambda_4,\tau\rangle_{(H_e^{1/2}(\{x_3=0\}))',H_e^{1/2}(\{x_3=0\})}\geq 0, \tag{5.24}$$

for all $[\mathbf{r},\varrho,\tau]\in\mathcal{C}(\hat{\mathbf{g}})\times\mathcal{C}(\hat{\phi}_1)\times\mathcal{C}(\hat{\phi}_2)$. Taking $\mathbf{r}=\mathbf{g}-\hat{\mathbf{g}}$, $\varrho=\phi_1-\hat{\phi}_1$ and $\tau=\phi_2-\hat{\phi}_2$ in (5.24), and recalling the definition of $\mathcal{B}_1$ given in (5.3), we obtain

$$\langle\gamma_4\hat{\mathbf{g}}+\boldsymbol{\lambda}_3,\mathbf{g}-\hat{\mathbf{g}}\rangle_{(H^{\frac12}(\Gamma_0^1))',H^{\frac12}(\Gamma_0^1)}+\langle\gamma_5\hat{\phi}_1+\lambda_2,\phi_1-\hat{\phi}_1\rangle_{(H^{\frac12}(\Gamma_0\setminus\{x_3=0\}))',H^{\frac12}(\Gamma_0\setminus\{x_3=0\})}$$
$$+\langle\gamma_6\hat{\phi}_2+\lambda_4,\phi_2-\hat{\phi}_2\rangle_{(H^{\frac12}(\{x_3=0\}))',H^{\frac12}(\{x_3=0\})}\geq 0.$$

Thus, the *optimality conditions* are

$$\langle\gamma_4\hat{\mathbf{g}}+\boldsymbol{\lambda}_3,\mathbf{g}-\hat{\mathbf{g}}\rangle_{(H^{\frac12}(\Gamma_0^1))',H^{\frac12}(\Gamma_0^1)}\;\geq\;0,\quad\forall\mathbf{g}\in\mathbf{U}_1, \tag{5.25}$$
$$\langle\gamma_5\hat{\phi}_1+\lambda_2,\phi_1-\hat{\phi}_1\rangle_{(H^{\frac12}(\Gamma_0\setminus\{x_3=0\}))',H^{\frac12}(\Gamma_0\setminus\{x_3=0\})}\;\geq\;0,\quad\forall\phi_1\in\mathcal{U}_2, \tag{5.26}$$
$$\langle\gamma_6\hat{\phi}_2+\lambda_4,\phi_2-\hat{\phi}_2\rangle_{(H^{\frac12}(\{x_3=0\}))',H^{\frac12}(\{x_3=0\})}\;\geq\;0,\quad\forall\phi_2\in\mathcal{U}_3. \tag{5.27}$$

Therefore, the state equations described in (2.1), the adjoint equations given in (5.16) and the optimality conditions obtained in (5.25)-(5.27), form the optimality system of the optimal control problem (5.1).

Remark 5.6. *Following the ideas of [14] we could suggest a semi-smooth Newton method applied to constrained boundary optimal control of the RBM system. However, due to the lack of sufficient regularity of the Lagrange multipliers for the pointwise control constraint in the optimality system, a direct application of the method to the infinite dimensional problem is not possible. Therefore, following the ideas of [14], seems reasonable to apply the semi-smooth Newton method to a regularization of the original control problem, and finally to analyze the convergence of the regularized solutions to the optimal solution.*

6 Second Order Sufficient Condition

In this section, we will analyze sufficient conditions for $\hat{\mathbf{z}}=[\hat{\mathbf{u}},\hat{\theta},\hat{\mathbf{g}},\hat{\phi}_1,\hat{\phi}_2]\in S_{ad}$ be a local optimal solution. We will establish a coercitivity condition on the second derivative of the Lagrangian $\mathcal{L}$ in order to assure that an admissible point $\hat{\mathbf{z}}$ is a local optimal solution. Here, we recall that

$$\mathcal{L}(\mathbf{z},\boldsymbol{\eta})\;=\mathcal{J}(\mathbf{z})-\langle\mathcal{F}_1(\mathbf{z}),\boldsymbol{\lambda}_1\rangle_{\mathbf{X}_0',\mathbf{X}_0}-\langle\mathcal{F}_2(\mathbf{z}),\lambda_2\rangle_{Y',Y}-\langle\boldsymbol{\lambda}_3,\mathcal{F}_3(\mathbf{z})\rangle_{(\widetilde{\mathbf{H}}_e^{1/2}(\Gamma_0))',\widetilde{\mathbf{H}}_e^{1/2}(\Gamma_0)}$$
$$-\langle\lambda_4,\mathcal{F}_4(\mathbf{z})\rangle_{(H_e^{1/2}(\{x_3=0\}))',H_e^{1/2}(\{x_3=0\})}.$$

We have that the Lagrange multiplier $\boldsymbol{\eta}$ satisfies $\mathcal{L}_{[\mathbf{u},\theta]}([\hat{\mathbf{z}},\boldsymbol{\eta}])[\mathbf{h}_1,h_2]=0$ for all $[\mathbf{h}_1,h_2]\in\widetilde{\mathbf{X}}\times H^1(\Omega)$, that is,

$$\gamma_1(\operatorname{rot}\hat{\mathbf{u}},\operatorname{rot}\mathbf{h}_1)_{L^2(\Omega)}+\gamma_2(\hat{\mathbf{u}}-\mathbf{u}_d,\mathbf{h}_1)_{L^2(\Omega)}+\gamma_3(\hat{\theta}-\theta_d,h_2)_{L^2(\Omega)}-Pr\,a(\mathbf{h}_1,\boldsymbol{\lambda}_1)-Pr\,M\,b_1(h_2,\boldsymbol{\lambda}_1)$$
$$-c(\hat{\mathbf{u}},\mathbf{h}_1,\boldsymbol{\lambda}_1)-c(\mathbf{h}_1,\hat{\mathbf{u}},\boldsymbol{\lambda}_1)+PrR(h_2,\lambda_{1_3})_{L^2(\Omega)}-c_1(\hat{\mathbf{u}},h_2,\lambda_2)-c_1(\mathbf{h}_1,\hat{\theta},\lambda_2)-a_1(h_2,\lambda_2)$$
$$-\langle Bh_2,\lambda_2\rangle_{\Gamma_1}-\langle\boldsymbol{\lambda}_3,\mathbf{h}_1|_{\Gamma_0}\rangle_{(\widetilde{\mathbf{H}}_e^{1/2}(\Gamma_0))',\widetilde{\mathbf{H}}_e^{1/2}(\Gamma_0)}-\langle\lambda_4,h_2|_{\{x_3=0\}}\rangle_{(H_e^{1/2}(\{x_3=0\}))',H_e^{1/2}(\{x_3=0\})}=0, \tag{6.1}$$

for all $[\mathbf{h}_1, h_2] \in \widetilde{\mathbf{X}} \times H^1(\Omega)$. In the next lemma we will establish a key estimate which is verified by the Lagrange multipliers $[\boldsymbol{\lambda}_1, \lambda_2] \in \mathbb{H}$.

Lemma 6.1. *Let $\hat{\mathbf{z}} = [\hat{\mathbf{u}}, \hat{\theta}, \hat{\mathbf{g}}, \hat{\phi}_1, \hat{\phi}_2]$ an admissible point for the constrained optimal control problem (2.2) and assume (5.5). Then, the Lagrange multipliers $[\boldsymbol{\lambda}_1, \lambda_2] \in \mathbb{H}$ satisfy*

$$\|\boldsymbol{\lambda}_1\|^2_{H^1(\Omega)} + \|\lambda_2\|^2_{H^1(\Omega)} \leq \frac{1}{\beta_0} C_1 \mathcal{M}[\hat{\mathbf{u}}, \hat{\theta}], \tag{6.2}$$

where $\mathcal{M}[\hat{\mathbf{u}}, \hat{\theta}] := \dfrac{\gamma_1^2}{Pr}\|\hat{\mathbf{u}}\|^2_{H^1(\Omega)} + \dfrac{\gamma_2^2}{Pr}\|\hat{\mathbf{u}} - \mathbf{u}_d\|^2_{L^2(\Omega)} + \gamma_3^2\|\hat{\theta} - \theta_d\|^2_{L^2(\Omega)}$ and C_1 is a positive constant depending only on Ω.

Proof: From (6.1), setting $[\mathbf{h}_1, h_2] = [\boldsymbol{\lambda}_1, \lambda_2] \in \mathbb{H}$, we have

$$\begin{aligned} Pr\|\nabla\boldsymbol{\lambda}_1\|^2_{L^2(\Omega)} + \|\nabla\lambda_2\|^2_{L^2(\Omega)} + B\|\lambda_2\|^2_{L^2(\Gamma_1)} &= \gamma_1(\mathrm{rot}\,\hat{\mathbf{u}}, \mathrm{rot}\,\boldsymbol{\lambda}_1)_{L^2(\Omega)} + \gamma_2(\hat{\mathbf{u}} - \mathbf{u}_d, \boldsymbol{\lambda}_1)_{L^2(\Omega)} \\ &+ \gamma_3(\hat{\theta} - \theta_d, \lambda_2)_{L^2(\Omega)} - Pr\, M\, b_1(\lambda_2, \boldsymbol{\lambda}_1) - c(\hat{\mathbf{u}}, \boldsymbol{\lambda}_1, \boldsymbol{\lambda}_1) - c(\boldsymbol{\lambda}_1, \hat{\mathbf{u}}, \boldsymbol{\lambda}_1) \\ &+ PrR(\lambda_2, \lambda_{1_3})_{L^2(\Omega)} - c_1(\hat{\mathbf{u}}, \lambda_2, \lambda_2) - c_1(\boldsymbol{\lambda}_1, \hat{\theta}, \lambda_2). \end{aligned} \tag{6.3}$$

Then, by using the Hölder, Poincaré and Young inequalities and Sobolev embeddings, from (6.3) we get

$$\begin{aligned} Pr\|\nabla\boldsymbol{\lambda}_1\|^2_{L^2(\Omega)} + \|\nabla\lambda_2\|^2_{L^2(\Omega)} &\leq \frac{C\gamma_1^2}{Pr}\|\hat{\mathbf{u}}\|^2_{H^1(\Omega)} + \frac{C\gamma_2^2}{Pr}\|\hat{\mathbf{u}} - \mathbf{u}_d\|^2_{L^2(\Omega)} + \frac{Pr}{2}\|\nabla\boldsymbol{\lambda}_1\|^2_{L^2(\Omega)} \\ &+ CPr(R+M)(\|\nabla\boldsymbol{\lambda}_1\|^2_{L^2(\Omega)} + \|\nabla\lambda_2\|^2_{L^2(\Omega)}) + C\|\nabla\boldsymbol{\lambda}_1\|^2_{L^2(\Omega)}\|\hat{\mathbf{u}}\|_{H^1(\Omega)} + C\gamma_3^2\|\hat{\theta} - \theta_d\|^2_{L^2(\Omega)} \\ &+ C\|\nabla\boldsymbol{\lambda}_1\|^2_{L^2(\Omega)}\|\hat{\theta}\|^2_{H^1(\Omega)} + \frac{1}{2}\|\nabla\lambda_2\|^2_{L^2(\Omega)}, \end{aligned}$$

where C only depends on Ω. Thus, we can get

$$\begin{aligned} &\left(Pr - C\left(Pr(M+R) + \|\hat{\mathbf{u}}\|_{H^1(\Omega)} + \|\hat{\theta}\|^2_{H^1(\Omega)}\right)\right)\|\nabla\boldsymbol{\lambda}_1\|^2_{L^2(\Omega)} \\ &+ \left(\frac{1}{2} - CPr(R+M)\right)\|\nabla\lambda_2\|^2_{L^2(\Omega)} \leq C\left(\frac{\gamma_1^2}{Pr}\|\hat{\mathbf{u}}\|^2_{H^1(\Omega)} + \frac{\gamma_2^2}{Pr}\|\hat{\mathbf{u}} - \mathbf{u}_d\|^2_{L^2(\Omega)} + \gamma_3^2\|\hat{\theta} - \theta_d\|^2_{L^2(\Omega)}\right). \end{aligned}$$

Then, since by hypothesis $\beta_0 = \min\left\{Pr - C\left(Pr(M+R) + \|\hat{\mathbf{u}}\|_{H^1(\Omega)} + \|\hat{\theta}\|^2_{H^1(\Omega)}\right), \frac{1}{2} - CPr(R+M)\right\} > 0$, using the Poincaré inequality we conclude (6.2). $\square$

Theorem 6.2. *Let $\hat{\mathbf{z}} = [\hat{\mathbf{u}}, \hat{\theta}, \hat{\mathbf{g}}, \hat{\phi}_1, \hat{\phi}_2]$ an admissible point for the constrained optimal control problem (2.2) and assume (5.5). If $\frac{C^2 C_1}{\beta_0 \Lambda^2}\mathcal{M}[\hat{\mathbf{u}}, \hat{\theta}] < 1$, where $\Lambda := \frac{C\min\{\gamma_4, \gamma_5, \gamma_6\}\beta_0^2}{(1+\beta_0)^2}$, and $\mathcal{M}[\hat{\mathbf{u}}, \hat{\theta}]$, C_1 are given in Lemma 6.1, then there exists $K_0 > 0$ such that*

$$\mathcal{L}_{\mathbf{zz}}[\hat{\mathbf{z}}, \boldsymbol{\eta}][\mathbf{t}, \mathbf{t}] \geq K_0\|\mathbf{t}\|^2_{\mathbb{G}}, \tag{6.4}$$

for all $\mathbf{t} \in \ker(\boldsymbol{F}_{\mathbf{z}}(\hat{\mathbf{z}}))$. Consequently, the point $\hat{\mathbf{z}}$ is a local optimal solution.

Proof: Let $\mathbf{t} = [\mathbf{h}_1, h_2, \mathbf{r}, \varrho, \tau] \in \mathbb{G}$. Then, the second derivative of the Lagrangian $\mathcal{L}$, with respect to $\mathbf{z}$ at the point $[\hat{\mathbf{z}}, \boldsymbol{\eta}]$ in all directions $[\mathbf{t}, \mathbf{t}]$, is given by

$$\begin{aligned} \mathcal{L}_{\mathbf{zz}}[\hat{\mathbf{z}}, \boldsymbol{\eta}][\mathbf{t}, \mathbf{t}] \;=\;& \gamma_1\|\mathrm{rot}\,\mathbf{h}_1\|^2_{L^2(\Omega)} + \gamma_2\|\mathbf{h}_1\|^2_{L^2(\Omega)} + \gamma_3\|h_2\|^2_{L^2(\Omega)} + \gamma_4\|\mathbf{r}\|^2_{H^{\frac{1}{2}}(\Gamma_0^1)} + \gamma_5\|\varrho\|^2_{H^{\frac{1}{2}}(\Gamma_0\setminus\{x_3=0\})} \\ &+ \gamma_6\|\tau\|^2_{H^{\frac{1}{2}}(\{x_3=0\})} - 2c(\mathbf{h}_1, \mathbf{h}_1, \lambda_1) - 2c_1(\mathbf{h}_1, h_2, \lambda_2). \end{aligned} \tag{6.5}$$

Thus, by using the Hölder and Young inequalities, we bound (6.5) as follows

$$\begin{aligned} \mathcal{L}_{\mathbf{zz}}[\hat{\mathbf{z}}, \hat{\eta}][\mathbf{t}, \mathbf{t}] \;\geq\;& \gamma_1\|\mathrm{rot}\,\mathbf{h}_1\|^2_{L^2(\Omega)} + \gamma_2\|\mathbf{h}_1\|^2_{L^2(\Omega)} + \gamma_3\|h_2\|^2_{L^2(\Omega)} + \gamma_4\|\mathbf{r}\|^2_{H^{\frac{1}{2}}(\Gamma_0^1)} + \gamma_5\|\varrho\|^2_{H^{\frac{1}{2}}(\Gamma_0\setminus\{x_3=0\})} \\ &+ \gamma_6\|\tau\|^2_{H^{\frac{1}{2}}(\{x_3=0\})} - C(\|\lambda_1\|_{H^1(\Omega)} + \|\lambda_2\|_{H^1(\Omega)})\|[\mathbf{h}_1, h_2]\|^2_{\widetilde{\mathbf{X}}\times H^1(\Omega)}. \end{aligned} \tag{6.6}$$

If $\mathbf{t} \in \ker(\boldsymbol{F}_{\mathbf{z}}(\hat{\mathbf{z}}))$, from (5.2)-(5.3) we obtain

$$Pr\, a(\mathbf{h}_1, \mathbf{v}) + Pr\, M\, b_1(h_2, \mathbf{v}) + c(\hat{\mathbf{u}}, \mathbf{h}_1, \mathbf{v}) + c(\mathbf{h}_1, \hat{\mathbf{u}}, \mathbf{v}) - PrR(h_2, v_3)_{L^2(\Omega)} = 0,\ \forall \mathbf{v} \in \mathbf{X}_0, \tag{6.7}$$

$$c_1(\hat{\mathbf{u}}, h_2, W) + c_1(\mathbf{h}_1, \hat{\theta}, W) + a_1(h_2, W) + \langle Bh_2, W\rangle_{\Gamma_1} - \langle \varrho, W\rangle_{\Gamma_0 \setminus \{x_3=0\}} = 0,\ \forall W \in Y, \tag{6.8}$$

$$\mathbf{h}_1|_{\Gamma_0} = \mathcal{B}_1 \mathbf{r}, \tag{6.9}$$

$$h_2|_{\{x_3=0\}} = \tau. \tag{6.10}$$

Proceeding as in the proof of Lemma 5.3, we can prove that there exist $[\mathbf{h}_1^\epsilon, h_2^\delta] \in \widetilde{\mathbf{X}} \times H^1(\Omega)$ such that $\mathbf{h}_1^\epsilon|_{\Gamma_0} = \mathcal{B}_1 \mathbf{r}$, $h_2^\delta\big|_{\{x_3=0\}} = \tau$, and the estimates in (3.8) remain true for $\theta_\delta = h_2^\delta$, $\phi_1 = \varrho$ and $\phi_2 = \tau$. Therefore, rewriting the unknowns $\mathbf{h}_1, h_2$ in the form $\mathbf{h}_1 = \mathbf{h}_1^\epsilon + \widetilde{\mathbf{h}}_1$, $h_2 = h_2^\delta + \widetilde{h}_2$ with $[\widetilde{\mathbf{h}}_1, \widetilde{h}_2] \in \mathbb{H}$ new unknown functions, from (6.7)-(6.10), we obtain

$$A([\widetilde{\mathbf{h}}_1, \widetilde{h}_2], [\mathbf{v}, W]) = \bar{I}[\mathbf{v}, W], \tag{6.11}$$

where A is the bilinear form defined in (5.12) and $\bar{I} : \mathbb{H} \to \mathbb{R}$ is defined by $\bar{I}[\mathbf{v}, W] := \langle \bar{\mathbf{a}}, \mathbf{v}\rangle + \langle \bar{b}, W\rangle$ with

$$\langle \bar{\mathbf{a}}, \mathbf{v}\rangle = -Pr\, a(\mathbf{h}_1^\epsilon, \mathbf{v}) - PrM\, b_1(h_2^\delta, \mathbf{v}) - c(\hat{\mathbf{u}}, \mathbf{h}_1^\epsilon, \mathbf{v}) - c(\mathbf{h}_1^\epsilon, \hat{\mathbf{u}}, \mathbf{v}) + PrR(h_2^\delta, v_3)_{L^2(\Omega)},$$

$$\langle \bar{b}, W\rangle = -c_1(\hat{\mathbf{u}}, h_2^\delta, W) - c_1(\mathbf{h}_1^\epsilon, \hat{\theta}, W) - a_1(h_2^\delta, W) - \left\langle Bh_2^\delta, W\right\rangle_{\Gamma_1} + \langle \varrho, W\rangle_{\Gamma_0 \setminus \{x_3=0\}}.$$

Moreover, arguing as in the proof of Lemma 5.3, from (6.11) we can get there exists $C > 0$, depending only on Pr, R, M, B, $\|[\hat{\mathbf{u}}, \hat{\theta}]\|_{\widetilde{\mathbf{X}} \times H^1(\Omega)}$ and Ω, such that

$$\|[\widetilde{\mathbf{h}}_1, \widetilde{h}_2]\|_{\mathbb{H}} \le \frac{C}{\beta_0}\left(\|[\mathbf{h}_1^\epsilon, h_2^\delta]\|_{\widetilde{\mathbf{X}} \times H^1(\Omega)} + \|\varrho\|_{H^{\frac{1}{2}}(\Gamma_0 \setminus \{x_3=0\})}\right),$$

with β_0 defined in (5.5), and consequently,

$$\begin{aligned} \|[\mathbf{h}_1, h_2]\|_{\widetilde{\mathbf{X}} \times H^1(\Omega)} &\le \|[\widetilde{\mathbf{h}}_1, \widetilde{h}_2]\|_{\mathbb{H}} + \|[\mathbf{h}_1^\epsilon, h_2^\delta]\|_{\widetilde{\mathbf{X}} \times H^1(\Omega)} \\ &\le C\left(\frac{1}{\beta_0} + 1\right)\left(\|\mathbf{r}\|_{H^{\frac{1}{2}}(\Gamma_0^1)} + \|\varrho\|_{H^{\frac{1}{2}}(\Gamma_0 \setminus \{x_3=0\})} + \|\tau\|_{H^{\frac{1}{2}}(\{x_3=0\})}\right). \end{aligned} \tag{6.12}$$

Thus, from (6.6) and (6.12) we get

$$\begin{aligned} \mathcal{L}_{\mathbf{zz}}[\hat{\mathbf{z}}, \boldsymbol{\eta}][\mathbf{t}, \mathbf{t}] \ &\ge\ \gamma_1 \|\mathrm{rot}\ \mathbf{h}_1\|^2_{L^2(\Omega)} + \gamma_2 \|\mathbf{h}_1\|^2_{L^2(\Omega)} + \gamma_3 \|h_2\|^2_{L^2(\Omega)} + \frac{\gamma_4}{2}\|\mathbf{r}\|^2_{H^{\frac{1}{2}}(\Gamma_0^1)} + \frac{\gamma_5}{2}\|\varrho\|^2_{H^{\frac{1}{2}}(\Gamma_0 \setminus \{x_3=0\})} \\ &\quad + \frac{\gamma_6}{2}\|\tau\|^2_{H^{\frac{1}{2}}(\{x_3=0\})} + C\frac{\min\{\gamma_4, \gamma_5, \gamma_6\}\beta_0^2}{(1+\beta_0)^2}\|[\mathbf{h}_1, h_2]\|^2_{\widetilde{\mathbf{X}} \times H^1(\Omega)} \\ &\quad - C(\|\lambda_1\|^2_{H^1(\Omega)} + \|\lambda_2\|^2_{H^1(\Omega)})\|[\mathbf{h}_1, h_2]\|^2_{\widetilde{\mathbf{X}} \times H^1(\Omega)} \\ &\ge\ \frac{\gamma_4}{2}\|\mathbf{r}\|^2_{H^{\frac{1}{2}}(\Gamma_0^1)} + \frac{\gamma_5}{2}\|\varrho\|^2_{H^{\frac{1}{2}}(\Gamma_0 \setminus \{x_3=0\})} + \frac{\gamma_6}{2}\|\tau\|^2_{H^{\frac{1}{2}}(\{x_3=0\})} \\ &\quad + \left(\Lambda - C(\|\lambda_1\|^2_{H^1(\Omega)} + \|\lambda_2\|^2_{H^1(\Omega)})\right)\|[\mathbf{h}_1, h_2]\|^2_{\widetilde{\mathbf{X}} \times H^1(\Omega)}. \end{aligned} \tag{6.13}$$

Therefore, by using estimate (6.2) in Lemma 6.1, from (6.13) we have

$$\begin{aligned} \mathcal{L}_{\mathbf{zz}}[\hat{\mathbf{z}}, \boldsymbol{\eta}][\mathbf{t}, \mathbf{t}] \ &\ge\ \frac{\gamma_4}{2}\|\mathbf{r}\|^2_{H^{\frac{1}{2}}(\Gamma_0^1)} + \frac{\gamma_5}{2}\|\varrho\|^2_{H^{\frac{1}{2}}(\Gamma_0 \setminus \{x_3=0\})} + \frac{\gamma_6}{2}\|\tau\|^2_{H^{\frac{1}{2}}(\{x_3=0\})} \\ &\quad + \left(\Lambda - C\left(\frac{C_1}{\beta_0}\mathcal{M}[\hat{\mathbf{u}}, \hat{\theta}]\right)^{1/2}\right)\|[\mathbf{h}_1, h_2]\|^2_{\widetilde{\mathbf{X}} \times H^1(\Omega)}. \end{aligned}$$

Then, since by hypothesis $C\left(\frac{C_1}{\beta_0}\mathcal{M}[\hat{\mathbf{u}},\hat{\theta}]\right)^{1/2} < \Lambda$, we deduce that $\Upsilon := \Lambda - C\left(\frac{C_1}{\beta_0}\mathcal{M}[\hat{\mathbf{u}},\hat{\theta}]\right)^{1/2} > 0$, and consequently,

$$\mathcal{L}_{\mathbf{zz}}[\hat{\mathbf{z}},\boldsymbol{\eta}][\mathbf{t},\mathbf{t}] \geq \min\left\{\Upsilon, \frac{\gamma_4}{2}, \frac{\gamma_5}{2}, \frac{\gamma_6}{2}\right\} \|\mathbf{t}\|^2_{\mathbb{G}}.$$

Thus, we conclude the coercitivity condition (6.4). Taking in particular $\mathbf{t} = [\mathbf{h}_1, h_2, \mathbf{r}, \varrho, \tau] \in \ker(\boldsymbol{F}_{\mathbf{z}}(\hat{\mathbf{z}}))$ with $[\mathbf{r}, \varrho, \tau] \in \mathcal{C}(\hat{\mathbf{g}}) \times \mathcal{C}(\hat{\phi}_1) \times \mathcal{C}(\hat{\phi}_2)$, we obtain that the point $\hat{\mathbf{z}}$ is a local optimal solution (cf. [32]). $\square$

7 Uniqueness of Optimal Solution

In this section we will establish a result related to the uniqueness of the optimal solution of problem (2.2). For that, suppose that there exist $\hat{\mathbf{z}}_i = [\hat{\mathbf{u}}_i, \hat{\theta}_i, \hat{\mathbf{g}}_i, \hat{\phi}^i_1, \hat{\phi}^i_2] \in \mathcal{S}_{ad}$ $(i = 1, 2)$ optimal solutions of problem (2.2), and let $\boldsymbol{\eta}_i = [\boldsymbol{\lambda}^i_1, \lambda^i_2, \boldsymbol{\lambda}^i_3, \lambda^i_4]$ two Lagrange multipliers corresponding to the solutions $\hat{\mathbf{z}}_i$ satisfying the relations (5.18), (5.20) and (5.25)-(5.27). Let us denote $\hat{\mathbf{u}} = \hat{\mathbf{u}}_1 - \hat{\mathbf{u}}_2$, $\hat{\theta} = \hat{\theta}_1 - \hat{\theta}_2$, $\hat{\mathbf{g}} = \hat{\mathbf{g}}_1 - \hat{\mathbf{g}}_2$, $\hat{\phi}_1 = \hat{\phi}^1_1 - \hat{\phi}^2_1$, $\hat{\phi}_2 = \hat{\phi}^1_2 - \hat{\phi}^2_2$, $\boldsymbol{\lambda}_1 = \boldsymbol{\lambda}^1_1 - \boldsymbol{\lambda}^2_1$, $\lambda_2 = \lambda^1_2 - \lambda^2_2$, $\boldsymbol{\lambda}_3 = \boldsymbol{\lambda}^1_3 - \boldsymbol{\lambda}^2_3$ and $\lambda_4 = \lambda^1_4 - \lambda^2_4$. Then, taking into account that $\hat{\mathbf{z}}_1$ and $\hat{\mathbf{z}}_2$ satisfy the equations (3.3)-(3.5), we deduce that $\hat{\mathbf{z}} = [\hat{\mathbf{u}}, \hat{\theta}, \hat{\mathbf{g}}, \hat{\phi}_1, \hat{\phi}_2]$ satisfies

$$Pr\, a(\hat{\mathbf{u}},\mathbf{v}) + Pr\, M\, b_1(\hat{\theta},\mathbf{v}) + c(\hat{\mathbf{u}}_1,\hat{\mathbf{u}},\mathbf{v}) + c(\hat{\mathbf{u}},\hat{\mathbf{u}}_2,\mathbf{v}) = \int_\Omega PrR\hat{\theta}v_3, \ \ \forall \mathbf{v} \in \mathbf{X}_0, \tag{7.1}$$

$$c_1(\hat{\mathbf{u}}_1,\hat{\theta},W) + c_1(\hat{\mathbf{u}},\hat{\theta}_2,W) + a_1(\hat{\theta},W) + \left\langle B\hat{\theta},W\right\rangle_{\Gamma_1} = \left\langle \hat{\phi}_1,W\right\rangle_{\Gamma_0\setminus\{x_3=0\}}, \ \ \forall W \in Y, \tag{7.2}$$

$$\hat{\mathbf{u}} = \hat{\mathbf{g}} \text{ on } \Gamma^1_0, \ \ \hat{\mathbf{u}} = \mathbf{0} \text{ on } \Gamma^2_0 \ \text{ and } \ \ \hat{\theta} = \hat{\phi}_2 \text{ on } \{x_3 = 0\}. \tag{7.3}$$

Proceeding as in the beginning of Subsection 3.2, we can prove that there exist $[\mathbf{u}_\epsilon, \theta_\delta] \in \widetilde{\mathbf{X}} \times H^1(\Omega)$ such that $\mathbf{u}_\epsilon|_{\Gamma^1_0} = \hat{\mathbf{g}}$, $\mathbf{u}_\epsilon|_{\partial\Omega\setminus\Gamma^1_0} = \mathbf{0}$, $\theta_\delta|_{\{x_3=0\}} = \hat{\phi}_2$, the estimates in (3.8) remain true for $\phi_1 = \hat{\phi}_1$ and $\phi_2 = \hat{\phi}_2$, and $\|\mathbf{u}_\epsilon\|_{H^1(\Omega)} \leq C\|\hat{\mathbf{g}}\|_{H^{\frac{1}{2}}(\Gamma^1_0)}$. Therefore, rewriting $[\hat{\mathbf{u}}, \hat{\theta}] \in \widetilde{\mathbf{X}} \times H^1(\Omega)$ in the form $\hat{\mathbf{u}} = \mathbf{u} + \mathbf{u}_\epsilon$ and $\hat{\theta} = \theta + \theta_\delta$, from (7.1)-(7.3) we obtain that $[\mathbf{u}, \theta] \in \mathbf{X}_0 \times Y$ satisfies

$$\begin{aligned} &Pr\, a(\mathbf{u},\mathbf{v}) + Pr\, M\, b_1(\theta,\mathbf{v}) + c(\hat{\mathbf{u}}_1,\mathbf{u},\mathbf{v}) + c(\mathbf{u},\hat{\mathbf{u}}_2,\mathbf{v}) = \int_\Omega PrR\theta v_3 + \int_\Omega PrR\theta_\delta v_3 \\ &\quad -Pr\, a(\mathbf{u}_\epsilon,\mathbf{v}) - Pr\, M\, b_1(\theta_\delta,\mathbf{v}) - c(\mathbf{u}_\epsilon,\hat{\mathbf{u}}_2,\mathbf{v}) - c(\hat{\mathbf{u}}_1,\mathbf{u}_\epsilon,\mathbf{v}), \ \ \forall \mathbf{v} \in \mathbf{X}_0, \end{aligned} \tag{7.4}$$

$$\begin{aligned} &c_1(\hat{\mathbf{u}}_1,\theta,W) + c_1(\mathbf{u},\hat{\theta}_2,W) + a_1(\theta,W) + \langle B\theta,W\rangle_{\Gamma_1} = \left\langle \hat{\phi}_1,W\right\rangle_{\Gamma_0\setminus\{x_3=0\}} \\ &\quad -c_1(\mathbf{u}_\varepsilon,\hat{\theta}_2,W) - c_1(\hat{\mathbf{u}}_1,\theta_\delta,W) - a_1(\theta_\delta,W) - \langle B\theta_\delta,W\rangle_{\Gamma_1} \ \forall W \in Y. \end{aligned} \tag{7.5}$$

Setting $\mathbf{v} = \mathbf{u}$ in (7.4), $W = \theta$ in (7.5), and using the Hölder inequality, Sobolev embeddings and the Poincaré inequality, we get

$$\begin{aligned} Pr\|\nabla\mathbf{u}\|_{L^2(\Omega)} &\leq C\left(\|\nabla\mathbf{u}\|_{L^2(\Omega)}\|\hat{\mathbf{u}}_2\|_{H^1(\Omega)} + \left(Pr + \|\hat{\mathbf{u}}_1\|_{H^1(\Omega)} + \|\hat{\mathbf{u}}_2\|_{H^1(\Omega)}\right)\|\mathbf{u}_\epsilon\|_{H^1(\Omega)}\right) \\ &\quad +C(PrR + PrM)\left(\|\nabla\theta\|_{L^2(\Omega)} + \|\theta_\delta\|_{H^1(\Omega)}\right), \end{aligned} \tag{7.6}$$

$$\begin{aligned} \|\nabla\theta\|_{L^2(\Omega)} &\leq C\left(\|\hat{\phi}_1\|_{H^{\frac{1}{2}}(\Gamma_0\setminus\{x_3=0\})} + \left(\|\nabla\mathbf{u}\|_{L^2(\Omega)} + \|\mathbf{u}_\epsilon\|_{H^1(\Omega)}\right)\|\hat{\theta}_2\|_{H^1(\Omega)}\right) \\ &\quad +C\left(1 + \|\hat{\mathbf{u}}_1\|_{H^1(\Omega)} + B\right)\|\theta_\delta\|_{H^1(\Omega)}. \end{aligned} \tag{7.7}$$

From inequality (3.25) we have that $\|\hat{\mathbf{u}}_i\|_{H^1(\Omega)} + \|\hat{\theta}_i\|_{H^1(\Omega)} \leq CS_i$, $i = 1, 2$, where

$$S_i = C\left(\|\mathbf{u}^0\|_{H^{\frac{1}{2}}(\Gamma^2_0)} + \|\hat{\mathbf{g}}_i\|_{H^{\frac{1}{2}}(\Gamma^1_0)} + \|\hat{\phi}^i_1\|_{H^{\frac{1}{2}}(\Gamma_0\setminus\{x_3=0\})} + \|\hat{\phi}^i_2\|_{H^{\frac{1}{2}}(\{x_3=0\})}\right). \tag{7.8}$$

Thus, from (7.6), (7.7) and (7.8) we obtain

$$\begin{aligned} Pr\|\nabla \mathbf{u}\|_{L^2(\Omega)} &\le C((S_2+Pr(R+M)S_2)\|\nabla \mathbf{u}\|_{L^2(\Omega)}+(Pr+S_1+S_2+Pr(R+M)S_2)\|\mathbf{u}_\epsilon\|_{H^1(\Omega)}) \\ &\quad +CPr(R+M)(1+S_1+B)\|\theta_\delta\|_{H^1(\Omega)}+CPr(R+M)\|\hat{\phi}_1\|_{H^{\frac12}(\Gamma_0\setminus\{x_3=0\})} \\ &\le C((S_2+Pr(R+M)S_2)\|\nabla \mathbf{u}\|_{L^2(\Omega)}+(Pr+S_1+S_2+Pr(R+M)S_2)\|\hat{\mathbf{g}}\|_{H^{\frac12}(\Gamma_0^1)}) \\ &\quad +CPr(R+M)(1+S_1+B)(\|\hat{\phi}_2\|_{H^{\frac12}(\{x_3=0\})}+\|\hat{\phi}_1\|_{H^{\frac12}(\Gamma_0\setminus\{x_3=0\})}). \end{aligned} \tag{7.9}$$

Taking Pr large enough and M, R small enough, from (7.9) we get

$$\|\nabla \mathbf{u}\|_{L^2(\Omega)} \le \mathcal{H}_0\left(\|\hat{\mathbf{g}}\|_{H^{\frac12}(\Gamma_0^1)}+\|\hat{\phi}_2\|_{H^{\frac12}(\{x_3=0\})}+\|\hat{\phi}_1\|_{H^{\frac12}(\Gamma_0\setminus\{x_3=0\})}\right), \tag{7.10}$$

where $\mathcal{H}_0 = C(Pr+S_1+S_2+B+1)(1+Pr(R+M))/Pr$, and therefore

$$\begin{aligned} \|\hat{\mathbf{u}}\|_{H^1(\Omega)} &\le \|\mathbf{u}\|_{H^1(\Omega)}+\|\mathbf{u}_\epsilon\|_{H^1(\Omega)} \le C\left(\|\nabla \mathbf{u}\|_{L^2(\Omega)}+\|\hat{\mathbf{g}}\|_{H^{\frac12}(\Gamma_0^1)}\right) \\ &\le C(\mathcal{H}_0+1)\left(\|\hat{\mathbf{g}}\|_{H^{\frac12}(\Gamma_0^1)}+\|\hat{\phi}_2\|_{H^{\frac12}(\{x_3=0\})}+\|\hat{\phi}_1\|_{H^{\frac12}(\Gamma_0\setminus\{x_3=0\})}\right). \end{aligned} \tag{7.11}$$

In the same spirit, from (7.7) and (7.10) we can obtain

$$\|\nabla \theta\|_{L^2(\Omega)} \le \mathcal{H}_1\left(\|\hat{\mathbf{g}}\|_{H^{\frac12}(\Gamma_0^1)}+\|\hat{\phi}_2\|_{H^{\frac12}(\{x_3=0\})}+\|\hat{\phi}_1\|_{H^{\frac12}(\Gamma_0\setminus\{x_3=0\})}\right),$$

where $\mathcal{H}_1 = C(S_1+S_2+B+1+S_2\mathcal{H}_0)$, and thus

$$\|\hat{\theta}\|_{H^1(\Omega)} \le C(\mathcal{H}_1+1)\left(\|\hat{\mathbf{g}}\|_{H^{\frac12}(\Gamma_0^1)}+\|\hat{\phi}_2\|_{H^{\frac12}(\{x_3=0\})}+\|\hat{\phi}_1\|_{H^{\frac12}(\Gamma_0\setminus\{x_3=0\})}\right). \tag{7.12}$$

On the other hand, subtracting equations (5.18) written for $\hat{\mathbf{u}}_i, \boldsymbol{\lambda}_1^i, \lambda_2^i, \hat{\theta}_i, \boldsymbol{\lambda}_3^i,\ i=1,2$, we have

$$\begin{aligned} &\gamma_1(\mathrm{rot}\ \hat{\mathbf{u}}, \mathrm{rot}\ \mathbf{h}_1)_{L^2(\Omega)}+\gamma_2(\hat{\mathbf{u}},\mathbf{h}_1)_{L^2(\Omega)} - Pr\, a(\mathbf{h}_1,\boldsymbol{\lambda}_1) - c(\hat{\mathbf{u}}_1,\mathbf{h}_1,\boldsymbol{\lambda}_1) - c(\hat{\mathbf{u}},\mathbf{h}_1,\boldsymbol{\lambda}_1^2) - c(\mathbf{h}_1,\hat{\mathbf{u}}_1,\boldsymbol{\lambda}_1) \\ &\quad -c(\mathbf{h}_1,\hat{\mathbf{u}},\boldsymbol{\lambda}_1^2) - c_1(\mathbf{h}_1,\hat{\theta}_1,\lambda_2) - c_1(\mathbf{h}_1,\hat{\theta},\lambda_2^2) - \langle \boldsymbol{\lambda}_3, \mathbf{h}_1|_{\Gamma_0}\rangle_{(\widetilde{\mathbf{H}}_e^{1/2}(\Gamma_0))',\widetilde{\mathbf{H}}_e^{1/2}(\Gamma_0)} = 0,\ \ \forall \mathbf{h}_1 \in \widetilde{\mathbf{X}}. \end{aligned} \tag{7.13}$$

Taking $\mathbf{h}_1 = \hat{\mathbf{u}}$ in (7.13) we obtain

$$\begin{aligned} &- Pr\, a(\hat{\mathbf{u}},\boldsymbol{\lambda}_1) - c(\hat{\mathbf{u}}_1,\hat{\mathbf{u}},\boldsymbol{\lambda}_1) - 2c(\hat{\mathbf{u}},\hat{\mathbf{u}},\boldsymbol{\lambda}_1^2) - c(\hat{\mathbf{u}},\hat{\mathbf{u}}_1,\boldsymbol{\lambda}_1) - c_1(\hat{\mathbf{u}},\hat{\theta}_1,\lambda_2) \\ &-c_1(\hat{\mathbf{u}},\hat{\theta},\lambda_2^2) - \langle \boldsymbol{\lambda}_3, \hat{\mathbf{u}}|_{\Gamma_0}\rangle_{(\widetilde{\mathbf{H}}_e^{1/2}(\Gamma_0))',\widetilde{\mathbf{H}}_e^{1/2}(\Gamma_0)} = -\gamma_1\|\mathrm{rot}\ \hat{\mathbf{u}}\|^2_{L^2(\Omega)} - \gamma_2\|\hat{\mathbf{u}}\|^2_{L^2(\Omega)}. \end{aligned} \tag{7.14}$$

Now, subtracting equations (5.20) written for $\hat{\mathbf{u}}_i, \boldsymbol{\lambda}_1^i, \lambda_2^i, \hat{\theta}_i, \lambda_4^i,\ i=1,2$, we get

$$\begin{aligned} &\gamma_3(\hat{\theta},h_2)_{L^2(\Omega)} - PrM\ b_1(h_2,\boldsymbol{\lambda}_1) + PrR\ (h_2,\lambda_{1_3})_{L^2(\Omega)} - c_1(\hat{\mathbf{u}}_1,h_2,\lambda_2) - c_1(\hat{\mathbf{u}},h_2,\lambda_2^2) \\ &-a_1(h_2,\lambda_2) - \langle Bh_2,\lambda_2\rangle_{\Gamma_1} - \langle \lambda_4, h_2|_{\{x_3=0\}}\rangle_{(H_e^{1/2}(\{x_3=0\}))',H_e^{1/2}(\{x_3=0\})} = 0,\ \ \forall h_2 \in H^1(\Omega). \end{aligned} \tag{7.15}$$

Taking $h_2 = \hat{\theta}$ in (7.15) we have

$$\begin{aligned} &- a_1(\hat{\theta},\lambda_2) - \langle B\hat{\theta},\lambda_2\rangle_{\Gamma_1} - \langle \lambda_4, \hat{\theta}|_{\{x_3=0\}}\rangle_{(H_e^{1/2}(\{x_3=0\}))',H_e^{1/2}(\{x_3=0\})} - c_1(\hat{\mathbf{u}}_1,\hat{\theta},\lambda_2) \\ &\quad -c_1(\hat{\mathbf{u}},\hat{\theta},\lambda_2^2) - PrM\ b_1(\hat{\theta},\boldsymbol{\lambda}_1) + PrR\ (\hat{\theta},\lambda_{1_3})_{L^2(\Omega)} = -\gamma_3\|\hat{\theta}\|^2_{L^2(\Omega)}. \end{aligned} \tag{7.16}$$

Moreover, setting $\mathbf{v} = \boldsymbol{\lambda}_1$ in (7.1) and $W = \lambda_2$ in (7.2), we obtain

$$Pr\, a(\hat{\mathbf{u}},\boldsymbol{\lambda}_1) + PrM\, b_1(\hat{\theta},\boldsymbol{\lambda}_1) + c(\hat{\mathbf{u}}_1,\hat{\mathbf{u}},\boldsymbol{\lambda}_1) + c(\hat{\mathbf{u}},\hat{\mathbf{u}}_2,\boldsymbol{\lambda}_1) = \int_\Omega PrR\hat{\theta}\lambda_{1_3}, \tag{7.17}$$

$$c_1(\hat{\mathbf{u}}_1,\hat{\theta},\lambda_2)+c_1(\hat{\mathbf{u}},\hat{\theta}_2,\lambda_2)+a_1(\hat{\theta},\lambda_2)+\left\langle B\hat{\theta},\lambda_2\right\rangle_{\Gamma_1}=\left\langle\hat{\phi}_1,\lambda_2\right\rangle_{\Gamma_0\backslash\{x_3=0\}}. \tag{7.18}$$

Adding (7.14), (7.16), (7.17), (7.18), and using (7.3) we have

$$\begin{aligned} &2c(\hat{\mathbf{u}},\hat{\mathbf{u}},\boldsymbol{\lambda}_1^2)+c(\hat{\mathbf{u}},\hat{\mathbf{u}},\boldsymbol{\lambda}_1)+c_1(\hat{\mathbf{u}},\hat{\theta},\lambda_2)+2c_1(\hat{\mathbf{u}},\hat{\theta},\lambda_2^2)\\ &+\langle\boldsymbol{\lambda}_3,\hat{\mathbf{g}}|_{\Gamma_0^1}\rangle_{(\tilde{\mathbf{H}}_e^{1/2}(\Gamma_0))',\tilde{\mathbf{H}}_e^{1/2}(\Gamma_0)}+\langle\lambda_4,\hat{\phi}_2|_{\{x_3=0\}}\rangle_{(H_e^{1/2}(\{x_3=0\}))',H_e^{1/2}(\{x_3=0\})}+\left\langle\hat{\phi}_1,\lambda_2\right\rangle_{\Gamma_0\backslash\{x_3=0\}}\\ &=\gamma_1\|\mathrm{rot}\ \hat{\mathbf{u}}\|^2_{L^2(\Omega)}+\gamma_2\|\hat{\mathbf{u}}\|^2_{L^2(\Omega)}+\gamma_3\|\hat{\theta}\|^2_{L^2(\Omega)}. \end{aligned} \tag{7.19}$$

Considering the optimality condition (5.25) $\langle\gamma_4\hat{\mathbf{g}}_i+\boldsymbol{\lambda}_3^i,\mathbf{g}-\hat{\mathbf{g}}_i\rangle_{(H^{\frac12}(\Gamma_0^1))',H^{\frac12}(\Gamma_0^1)}\geq 0$, $i=1,2$, taking $\mathbf{g}=\hat{\mathbf{g}}_1$ at $i=2$, $\mathbf{g}=\hat{\mathbf{g}}_2$ at $i=1$, and adding these relations, we obtain

$$\langle\boldsymbol{\lambda}_3,\hat{\mathbf{g}}|_{\Gamma_0^1}\rangle_{(H^{\frac12}(\Gamma_0^1))',H^{\frac12}(\Gamma_0^1)}\leq-\gamma_4\|\hat{\mathbf{g}}\|^2_{H^{\frac12}(\Gamma_0^1)}. \tag{7.20}$$

Analogously, from the optimality conditions (5.26)-(5.27) we get

$$\left\langle\hat{\phi}_1,\lambda_2\right\rangle_{(H^{\frac12}(\Gamma_0\backslash\{x_3=0\}))',H^{\frac12}(\Gamma_0\backslash\{x_3=0\})}\leq-\gamma_5\|\hat{\phi}_1\|^2_{H^{\frac12}(\Gamma_0\backslash\{x_3=0\})}, \tag{7.21}$$

$$\langle\lambda_4,\hat{\phi}_2|_{\{x_3=0\}}\rangle_{(H^{\frac12}(\{x_3=0\}))',H^{\frac12}(\{x_3=0\})}\leq-\gamma_6\|\hat{\phi}_2\|^2_{H^{\frac12}(\{x_3=0\})}. \tag{7.22}$$

By using (7.11),(7.12) and (6.2) we can bound the four terms of the left hand side of (7.19) as follows

$$2c(\hat{\mathbf{u}},\hat{\mathbf{u}},\boldsymbol{\lambda}_1^2)\leq C(\mathcal{H}_0+1)^2\frac{1}{\beta_0^{1/2}}(\mathcal{M}[\hat{\mathbf{u}}_1,\hat{\theta}_1])^{1/2}\left(\|\hat{\mathbf{g}}\|^2_{H^{\frac12}(\Gamma_0^1)}+\|\hat{\phi}_2\|^2_{H^{\frac12}(\{x_3=0\})}+\|\hat{\phi}_1\|^2_{H^{\frac12}(\Gamma_0\backslash\{x_3=0\})}\right), \tag{7.23}$$

$$\begin{aligned} c(\hat{\mathbf{u}},\hat{\mathbf{u}},\boldsymbol{\lambda}_1)\ &\leq\ C(\mathcal{H}_0+1)^2\frac{1}{\beta_0^{1/2}}((\mathcal{M}[\hat{\mathbf{u}}_1,\hat{\theta}_1])^{1/2}+(\mathcal{M}[\hat{\mathbf{u}}_2,\hat{\theta}_2])^{1/2})\\ &\times\left(\|\hat{\mathbf{g}}\|^2_{H^{\frac12}(\Gamma_0^1)}+\|\hat{\phi}_2\|^2_{H^{\frac12}(\{x_3=0\})}+\|\hat{\phi}_1\|^2_{H^{\frac12}(\Gamma_0\backslash\{x_3=0\})}\right), \end{aligned} \tag{7.24}$$

$$\begin{aligned} c_1(\hat{\mathbf{u}},\hat{\theta},\lambda_2)\ &\leq\ C(\mathcal{H}_0+1)(\mathcal{H}_1+1)\frac{1}{\beta_0^{1/2}}((\mathcal{M}[\hat{\mathbf{u}}_1,\hat{\theta}_1])^{1/2}+(\mathcal{M}[\hat{\mathbf{u}}_2,\hat{\theta}_2])^{1/2})\\ &\times\left(\|\hat{\mathbf{g}}\|^2_{H^{\frac12}(\Gamma_0^1)}+\|\hat{\phi}_2\|^2_{H^{\frac12}(\{x_3=0\})}+\|\hat{\phi}_1\|^2_{H^{\frac12}(\Gamma_0\backslash\{x_3=0\})}\right), \end{aligned} \tag{7.25}$$

$$\begin{aligned} 2c_1(\hat{\mathbf{u}},\hat{\theta},\lambda_2^2)\ &\leq\ C(\mathcal{H}_0+1)(\mathcal{H}_1+1)\frac{1}{\beta_0^{1/2}}(\mathcal{M}[\hat{\mathbf{u}}_2,\hat{\theta}_2])^{1/2}\\ &\times\left(\|\hat{\mathbf{g}}\|^2_{H^{\frac12}(\Gamma_0^1)}+\|\hat{\phi}_2\|^2_{H^{\frac12}(\{x_3=0\})}+\|\hat{\phi}_1\|^2_{H^{\frac12}(\Gamma_0\backslash\{x_3=0\})}\right). \end{aligned} \tag{7.26}$$

From (7.19) and taking into account estimates (7.20)-(7.26) we obtain

$$\begin{aligned} &\gamma_1\|\mathrm{rot}\ \hat{\mathbf{u}}\|^2_{L^2(\Omega)}+\gamma_2\|\hat{\mathbf{u}}\|^2_{L^2(\Omega)}+\gamma_3\|\hat{\theta}\|^2_{L^2(\Omega)}\leq(\mathcal{I}-\min\{\gamma_4,\gamma_5,\gamma_6\})\\ &\times\left(\|\hat{\mathbf{g}}\|^2_{H^{\frac12}(\Gamma_0^1)}+\|\hat{\phi}_2\|^2_{H^{\frac12}(\{x_3=0\})}+\|\hat{\phi}_1\|^2_{H^{\frac12}(\Gamma_0\backslash\{x_3=0\})}\right), \end{aligned} \tag{7.27}$$

where $\mathcal{I}:=C\max\{(\mathcal{H}_0+1)^2,(\mathcal{H}_0+1)(\mathcal{H}_1+1)\}\frac{1}{\beta_0^{1/2}}((\mathcal{M}[\hat{\mathbf{u}}_1,\hat{\theta}_1])^{1/2}+(\mathcal{M}[\hat{\mathbf{u}}_2,\hat{\theta}_2])^{1/2})$. By assuming Pr large enough and R,M small enough such that $\mathcal{H}_0$ be small enough and $\mathcal{I}<\min\{\gamma_4,\gamma_5,\gamma_6\}$, from (7.27) we obtain

$$\|\hat{\mathbf{u}}\|_{L^2(\Omega)}+\|\hat{\theta}\|_{L^2(\Omega)}+\|\hat{\mathbf{g}}\|_{H^{\frac12}(\Gamma_0^1)}+\|\hat{\phi}_2\|_{H^{\frac12}(\{x_3=0\})}+\|\hat{\phi}_1\|_{H^{\frac12}(\Gamma_0\backslash\{x_3=0\})}=0,$$

which implies that $\hat{\mathbf{z}}_1=\hat{\mathbf{z}}_2$. Thus, we have proved the following theorem:

Theorem 7.1. *If Pr is large enough and R, M are small enough, then the optimal solution of problem (2.2) is unique.*

References

[1] F. Abergel and F. Casas. Some optimal control problems of multistate equations appearing in fluid mechanics. *RAIRO Modél. Math. Anal. Numér.,* 27 (2) (1993), 223-247.

[2] G.V. Alekseev. Solvability of stationary boundary control problems for heat convection equations. *Siberian Math. J.,* 39 (1998), 982-998.

[3] G.V. Alekseev and E.A. Adomavichus. Theoretical analysis of inverse extremal problems of admixture diffusion in viscous fluids. *J. Inverse Ill-Posed Probl.,* 9 (2001), 435-468.

[4] G.V. Alekseev and D.A. Tereshko. Extremum problems of boundary control for steady equations of thermal convection. *J. Appl. Mech. Tech. Phys.,* 51 (2010), 510-520.

[5] H. Bénard. Les tourbillons cellulaires dans une nappe liquide. *Revue Gén. Sci. Pures Appl.,* 11 (1900), 1261-1271 and 1309-1328.

[6] M.J. Block. Surface tension as the cause of Bénard cells and surface deformation of a liquid film. *Discrete Dyn. Nat. Soc.,* 178 (1956), 650-651.

[7] S. Chandrasekhar. Hydrodynamic and Hydromagnetic Stability. *Oxford University Press.,* (1961).

[8] M.G. Crandall and P.H. Rabinowitz. Bifurcation from simple eigenvalues. *J. Funct. Anal.,* 8 (1971), 321-340.

[9] P.C. Dauby and G. Lebon. Bénard-Marangoni instability in rigid rectangular containers. *J. Fluid Mech.,* 329 (1996), 25-64.

[10] M. Dauge. Stationary Stokes and Navier-Stokes systems on two- and three-dimensional with corners. Part 1: Linearized equations, *SIAM J. Math. Anal.,* 20 (1989), 74-97.

[11] M. Dauge. Problèmes mixtes pour le laplacien dans des domaines polyhédraux courbes. *C. R. Acad. Sci. Paris Sér. I Math.,* 309 (1989), 553-558.

[12] M. Dauge. Neumann and mixed problems on curvilinear polyhedra. *Integral Equations Operator Theory.*, 15 (1992), 227-261.

[13] R. Dautray and J.L. Lions. Mathematical Analysis and Numerical Methods for Science and Technology, Vol. 2, Springer, Berlin, 1988.

[14] J.C. De los Reyes and K. Kunisch. A semi-smooth Newton method for control constrained boundary optimal control of the Navier-Stokes equations. *Nonlinear Anal.,* 62, (2005), 1289-1316.

[15] V. Girault and P.-A. Raviart. Finite Element Methods for Navier-Stokes Equations. Springer-Verlag, Berlin, (1986).

[16] P. Grisvard. Elliptic Problems in Nonsmooth Domains. Monogr. Stud. in Math., 24, Boston, MA, (1985).

[17] G.P. Galdi. An introduction to the mathematical theory of the Navier-Stokes equations. Vol. I, Springer-Verlag, New York, (1994).

[18] G.P. Galdi. An introduction to the mathematical theory of the Navier-Stokes equations. Vol. II, Springer-Verlag, New York, (1994).

[19] M.D. Gunzburger and S. Manservisi. Analysis and approximation of the velocity tracking problem for Navier-Stokes flows with boundary control, *SIAM J. Control Optim.,* 39 (2000), 594-634.

[20] M. Heinkenschloss. Formulation and analysis of sequential quadratic programming method for the optimal Dirichlet boundary control of Navier-Stokes flow, Optimal Control, Gainesville, FL, 1997, Kluwer Academic Publishers, Dordrecht, 1998, pp. 178-203.

[21] S. Hoyas. Estudio teórico y numérico de un problema de convección de Bénard-Marangoni en un anillo. Ph.D thesis, U. Complutense de Madrid, (2003).

[22] S. Hoyas, H. Herrero and A. M. Mancho. Thermal convection in a cylindrical annulus heated laterally. *J. Phys. A,* 35 (2002), 4067-4083.

[23] C. Hu, R. Temam and M. Ziane. Regularity results for linear elliptic problems related to the primitive equations. Dedicated to the memory of Jacques-Louis Lions. *Chinese Ann. Math.,* 23 (2002), 277-292.

[24] R.B. Kellogg and J. Osborn. *A regularity result for the Stokes problem in a convex polygon.* J. Funct. Anal., 21 (1976), 397-431.

[25] A.M. Kvarving, T. Bjontegaard and E.M. Ronquist. On pattern selection in three-dimensional Bénard-Marangoni flows. *Commun. Comput. Phys.,* 11 (2012), 893-924.

[26] M. Lappa. Thermal Convection: Patterns, Evolution and Stability Naples. Wiley, (2010).

[27] H.-C. Lee and O.Y. Imanuvilov. Analysis of Neumann boundary optimal control problems for the stationary Boussinesq equations including solid media. *SIAM J. Control Optim.,* 39 (2000), 457-477.

[28] H.-C. Lee and S. Kim. Finite element approximation and computations of optimal Dirichlet boundary control problems for the Boussinesq equations. *J. Korean Math. Soc.,* 41 (2004), 681-715.

[29] H.-C. Lee. Optimal control problems for the two dimensional Rayleigh-Bénard type convection by a gradient method. *Japan J. Indust. Appl. Math.,* 26 (2009), 93-121.

[30] E. Feireisl and A. Novotný. Singular limits in thermodynamics of viscous fluids, Birkhauser, Basel, 2009.

[31] C. Marangoni. Sull'espansione delle gocce di un liquido galleggianti sulla superficie di altro liquido. *Ann. Phys. Chem.,* 143 (1871), 337-354.

[32] H. Maurer. First and second order sufficient optimality conditions in mathematical programming and optimal control. *Math. Program. Study,* 14 (1981), 163-177.

[33] V. Maz'ya and J. Rossmann. L^p estimates of solutions to mixed boundary value problems for the Stokes system in polyhedral domains. *Math. Nachr.,* 280 (2007), 751-793.

[34] V. Maz'ya and J. Rossmann. Mixed boundary value problems for the stationary Navier-Stokes system in polyhedral domains. *Arch. Ration. Mech. Anal.,* 194 (2009), 669-712.

[35] R. Pardo, S. Herrero and S. Hoyas. Theoretical study of a Bénard–Marangoni problem. *J. Math. Anal. Appl.* 376 (2011), 231-246.

[36] J.R.A. Pearson. On convection cells induced by surface tension. *J. Fluid Mech.,* 4 (1958), 489.

[37] L. Rayleigh. On convection currents in horizontal layer of fluid when the higher temperature is on the under side. *Philos. Stud.,* 192 (1916), 529-546.

[38] M.A. Rodríguez-Bellido, M.A. Rojas-Medar and E.J. Villamizar-Roa. The Boussinesq system with mixed nonsmooth boundary data. *C. R. Math. Acad. Sci. Paris.,* 343 (2006), 191-196.

[39] E.J. Villamizar-Roa, M.A. Rodríguez-Bellido and M.A. Rojas-Medar. Some properties of a class of abstract stationary equations. *Nonlinear Anal.,* 64 (2006), 2203-2214.

[40] J. Zowe and S. Kurcyusz. Regularity and Stability for the Mathematical Programming Problem in Banach Spaces. Appl. Math. Optim., 5 (1979), 49-62.